Université Toulouse III - Paul Sabatier

Stochastic calculus application in quantum mechanics

by

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"Quantum theory provides us with a striking illustration of the fact that we can fully understand a connection though we can only speak of it in images and parables."

Werner Heisenberg

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Abstract

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Internship Report

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In this report we are interested in the statistic formulation of Schrödinger equation in quantum mechanics. The first chapter introduces relevant background theory and some mathematical tools, for example, Itô formula and SDE. In the second chapter, we study the linear and non-linear formulation of Schrödinger equation in quantum mechanics and collect some heuristic explanations for them.

Acknowledgements

We do this report mainly for understanding exsiting results from others but not stating new discoveries. The first chapter is taken from Daniel Revuz and Marc Yor's [4]. The second chapter is taken from Alberto Barchielli and Matteo Gregoratti's [1]. If there are results without explicit citations, it is not our creation but negligence.

Our project is under the guide of Tristan Benoist and Clément Pellegrini. They introduce us this topic and offer valuable indications when we are confused. Whithout their help, we could hardly archieve anything.

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Chapter 1

Background theory

1.1 Wiener process

The goal of the theory of stochastic processes is to construct and study mathematical models of physical systems which evolve in time according to a random mechanism. Thus, a stochastic process will be a family of random variables indexed by time.

Definition 1.1 (Stochastic process)**.** Let T be a set, $(E, \mathcal{E})$ a measurable space. A stochastic process indexed by T, taking its values in $(E, \mathcal{E})$, is a family of measurable mappings $X_t, t \in T$, from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into $(E, \mathcal{E})$. The space $(E, \mathcal{E})$ is called the state space.

Definition 1.2. Let E be a topological space and $\mathcal{E}$ the σ-algebra of its Borel subsets. A process X with values in $(E, \mathcal{E})$ is said to be *a.s.* continuous if, for almost all ω, the function $t \mapsto X_t(\omega)$ is continuous.

In our discussion we deal with the case $T = \mathbb{R}_+ := [0, +\infty[$ and E will usually be $\mathbb{R}^d$ and $\mathcal{E}$ the Borel σ-algebra on E. An n-dimensional complex state space is always identified with a $2n$-dimensional real state space.

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration is a family $(\mathcal{F}_t)_{t\geq 0}$ of increasing sub-σ-algebras of $\mathcal{F}$, i.e., $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $0 \leq s < t < +\infty$. Sometimes, $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ is said to be a stochastic basis. Typically, a filtration describes the accumulation of information during time: each $\mathcal{F}_t$ is the collection of all the events which we can decide whether they have been verified or not up to time t.

Let us denote by $\mathcal{N}$ the class of all $\mathbb{P}$-null sets in $\mathcal{F}$, i.e.,

$$\mathcal{N} := \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$$

Definition 1.3. The filtration is said to be right continuous if $\mathcal{F}_t = \mathcal{F}_{t_+}$ for all $t \geq 0$, where $\mathcal{F}_{t_+}$ is the σ-algebra of events decidable immediately after t, i.e.,

$$\mathcal{F}_{t_+} := \bigcap_{s:s>t} \mathcal{F}_s.$$

The stochastic basis (or the filtration) is said to satisfy the usual conditions if the filtration is right continuous and $\mathcal{F}_0$ contains $\mathcal{N}$. Obviously $\mathcal{N} \subset \mathcal{F}_0$ implies $\mathcal{N} \subset \mathcal{F}_t, \forall t \geq 0$.

Definition 1.4. A process X is adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

A Wiener process W is a Gaussian process, that is: for any sequence $0 = t_0 < t_1 < \ldots < t_n$, the vector r.v. $(W_{t_0}, \ldots, W_{t_n})$ is a vector Gaussian r.v., with independent and stationary increments (which will be clear later), with mean zero and variance proportional to t, or covariance matrix proportional to $t\mathbb{1}$ ($\mathbb{1}$ means an identity matrix) in the multidimensional case. It is usual to take it exactly equal to $t\mathbb{1}$ (standard Wiener process). At the price of a modification, it is always possible to obtain continuous trajectories. Moreover, for the developments of stochastic calculus, where adapted processes are integrated with respect to Wiener processes, it is convenient to include the filtration in the definition of Wiener process. Without loss of generality, we have the following definition.

Definition 1.5. Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a stochastic basis. A d-dimensional Wiener process $W \equiv \{W_j(t), t \geq 0, j = 1, \ldots, d\}$ is a continuous, $\mathbb{R}^d$-valued, adapted process with the following properties:

1. $W(0) = 0$ *a.s.*;
2. for $0 \leq s < t < +\infty$ the increment $W(t) - W(s)$ is normal with vector of means 0 and covariance matrix $(t - s)\mathbb{1}$;
3. for $0 \leq s < t < +\infty$ the increment $W(t) - W(s)$ is independent of $\mathcal{F}_s$.

Remark 1.6. It would be equivalent to define a one-dimensional Wiener process and to say that a d-dimensional Wiener process is a collection of d independent one-dimensional Wiener processes. So for most of the cases, discussion of a one-dimensional Wiener is already enough for us.

From now on, we shall usually denote by W_t the time t slice of a one-dimensional Wiener process and by $W_j(t)$ the time t slice of the jth component of a multidimensional Wiener process. One can tell from the notation whether we are talking about Wiener process as multidimensional process or not.

For the existence of Wiener process, we restate Theorem (1.8) Chpater I, p.19 in [4] here:

Theorem 1.7. *There exists an almost surely continuous process W with independent increments such that for each t, the random variable W_t is centered, Gaussian and has variance t.*

The properties stated in Theorem (1.7) imply those we already know. For instance, for $s < t$, the increments $W_t - W_s$ are Gaussian centered with variance $t - s$; indeed, we can write

$$W_t = W_s + (W_t - W_s)$$

and using the independence of W_s and $W_t - W_s$, we get, by considering characteristic functions,

$$\exp\left(-\frac{tu^2}{2}\right) = \exp\left(-\frac{su^2}{2}\right) E\left[\exp\left(\mathrm{i}u\left(W_t - W_s\right)\right)\right]$$

whence $E\left[\exp\left(\mathrm{i}u\left(W_t - W_s\right)\right)\right] = \exp\left(-\frac{(t-s)}{2}u^2\right)$ follows. We have an equivalence in the theorem between assertions $\mathrm{var}(W_t) = t$ and $\mathrm{cov}(W_t, W_s) = \min(t, s)$. Indeed, if as proven above (supposing $t > s$) $\mathrm{var}(W_t - W_s) = t - s$, i.e., $E[(W_t - W_s)^2)] = t - s$ we then have $E[W_t^2 + W_s^2] - 2E(W_t W_s) = \mathrm{var}(W_t) + \mathrm{var}(W_s) - 2\,\mathrm{cov}(W_t, W_s) = t + s - 2\,\mathrm{cov}(W_t, W_s) = t - s$; so $\mathrm{cov}(W_t, W_s) = \min(s, t) = s$.

By discarding a negligible set, we may, and often will, consider that all paths of W are continuous.

1.2 Martingale and Quadratic Variations

Definition 1.8. Let X be a process adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ and each X_t be integrable. We say that X is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$, if $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ for all $0 \leq s \leq t$.

Then a martingale is a stochastic process where the prediction of the trajectory at the time t with respect to the past time before s is simply the trajectory at the time s. We can interpret $\mathbb{E}\left[X(t)|\mathcal{F}_s\right]$ by saying that past and present (the present is s) are frozen and we take the mean of $X(t)$ only with respect to all the stochasticity entering into play in the future.

Definition 1.9. Given a filtration $(\mathcal{F}_t)_{t\geq 0}$, a random variable $\tau : \Omega \to [0, +\infty]$ is called a stopping time, or, better, an $(\mathcal{F}_t)$-stopping time, if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

A stopping time describes the occurrence instant of a random phenomenon observed during the random experiment related to $(\mathcal{F}_t)$. An example in real life might be the time at which a gambler leaves the gambling table, which might be a function of their previous winnings (for example, he might leave only when he goes broken), but he can't choose to go or stay based on the outcome of games that haven't been played yet.

Definition 1.10. A process X is called measurable if the function $[0, +\infty) \times \Omega \ni (t, \omega) \mapsto X(t, \omega)$ is $\mathcal{B}([0 + \infty)) \otimes \mathcal{F}$-measurable.

We need the joint measurability in t and ω, for instance, when we want to exchange an integral over time and an expectation by invoking Fubini theorem.

Definition 1.11. A process X is called progressively measurable or progressive if for every $T \geq 0$ the function $[0, T] \times \Omega \ni (t, \omega) \mapsto X(t, \omega)$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$-measurable.

Trivially, a progressive process is adapted and measurable.

Remark 1.12. If τ is a finite stopping time and X is a measurable process, then $\omega \mapsto X(\tau(\omega), \omega)$ is a random variable. In this statement the joint measurability in (t, ω) of X is crucial in order that $X(\tau)$ be $\mathcal{F}$-measurable. Moreover, if τ is a stopping time, then $\tau \wedge t := \min\{\tau, t\}$ is a finite stopping time and, if X is a progressive process, $X(t \wedge \tau)$ is an $\mathcal{F}_t$-measurable random variable and the *stopped process* $\{X(t \wedge \tau), t \geq 0\}$, usually written as X^τ, is a progressive process. Again the progressive character of X is crucial in order that the stopped process be adapted.

Proposition 1.13. *If M is a continuous martingale and T a stopping time, the stopped process M^T, i.e., $\{M(t \wedge T), t \geq 0\}$ is a martingale with respect to $(\mathcal{F}_t)$.*

Proof. The process M^T is obviously continuous and adapted. Firstly we use a weak form of optional stopping theorem, saying that a martingale has equal expectation at any bounded stopping time. If S is a bounded stopping time, so is $S \wedge T$; hence

$$\mathbb{E}\left[M_S^T\right] = \mathbb{E}\left[M_{S \wedge T}\right] = \mathbb{E}\left[M_0\right] = \mathbb{E}\left[M_0^T\right].$$

Then we use this conclusion twice to get our desired equality. If $s < t$ and $A \in \mathcal{F}_s$ the r.v. $T = t\mathbf{1}_{A^c} + s\mathbf{1}_A$ is a stopping time and consequently

$$\mathbb{E}\left[X_0\right] = \mathbb{E}\left[X_T\right] = \mathbb{E}\left[X_t \mathbf{1}_{A^c}\right] + \mathbb{E}\left[X_s \mathbf{1}_A\right].$$

On the other hand, t itself is a stopping time, and

$$\mathbb{E}\left[X_0\right] = \mathbb{E}\left[X_t\right] = \mathbb{E}\left[X_t \mathbf{1}_{A^c}\right] + \mathbb{E}\left[X_t \mathbf{1}_A\right].$$

Comparing the two equalities yields $X_s = \mathbb{E}[X_t|\mathcal{F}_s]$. □

Definition 1.14. A process X is a local martingale, with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$, if there exists an increasing sequence of stopping times τ_n such that $\tau_n \underset{n\to+\infty}{\longrightarrow} +\infty$ *a.s.* and $\left\{X_{(t\wedge\tau_n)}, t \geq 0\right\}$ is an $(\mathcal{F}_t)$-martingale for all n.

Definition 1.15. A process A is of finite variation if it is adapted and the paths $t \to A_t(\omega)$ are finite, continuous and of finite variation for almost every ω.

Proposition 1.16. *A continuous martingale M cannot be of finite variation unless it is constant.*

Proof. We may suppose that $M_0 = 0$ and prove that M is identically zero if it is of finite variation. Let V_t be the variation of M on $[0, t]$ and define a stopping time

$$S_n = \inf\{s : V_s \geq n\};$$

then the martingale (by Proposition 1.13) M^{S_n}, i.e., the stopped process $\{M(t\wedge S_n), t \geq 0\}$, is of bounded variation. Thus, it is enough to prove the result whenever the variation of M is bounded by a number K.

Let $\Delta = \{t_0 = 0 < t_1 < \ldots < t_k = t\}$ be a subdivision of $[0, t]$; we have

$$\begin{aligned}
\mathbb{E}\left[M_t^2\right] &= \mathbb{E}\left[\sum_{i=0}^{k-1}\left(M_{t_{i+1}}^2 - M_{t_i}^2\right)\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{k-1}\left(M_{t_{i+1}} - M_{t_i}\right)^2\right]
\end{aligned}$$

since M is a martingale. As a result,

$$\mathbb{E}\left[M_t^2\right] \leq \mathbb{E}\left[V_t\left(\sup_i\left|M_{t_{i+1}} - M_{t_i}\right|\right)\right] \leq K\mathbb{E}\left[\sup_i\left|M_{t_{i+1}} - M_{t_i}\right|\right]$$

when the modulus of Δ goes to zero, this quantity goes to zero since M is continuous, hence $M = 0$ *a.s.*. □

Because of this proposition, we will not be able to define integrals with respect to M by a path by path procedure as Stieltjes integral. We will have to use a global method in which the notions we are about to introduce play a crucial role. If $\Delta = \{t_0 = 0 < t_1 < \ldots\}$ is a subdivision of $\mathbb{R}_+$ with only a finite number of points in each interval $[0, t]$ we define, for a process X

$$T_t^\Delta(X) = \sum_{i=0}^{k-1}\left(X_{t_{i+1}} - X_{t_i}\right)^2 + \left(X_t - X_{t_k}\right)^2$$

where k is such that $t_k \leq t < t_{k+1}$; we will write simply T_t^Δ if there is no risk of confusion.

Definition 1.17. X is said to be of *finite quadratic variation* if there exists a process $\langle X, X\rangle$ such that for each t, T_t^Δ converges in probability to $\langle X, X\rangle_t$ as the modulus of Δ on $[0,t]$, i.e., $\sup_i |t_{i+1}-t_i|$, goes to zero.

To be concrete, we caculate quadratic variation for one-dimensional Wiener process.

Proposition 1.18. *Let W be a one-dimensional Wiener process (also called Brownian motion in mathematical literature), we have $\langle W, W\rangle_t = t$.*

Proof. Let $\Delta = \{t_0 = 0 < t_1 < \ldots < t_n = t\}$ be a subdivision of $[0,t]$, and define $X_i = W_{t_i} - W_{t_{i-1}}$ for $i = 1, 2, \ldots, n$. We should have $\{X_i\}$ are independent and $\mathbb{E}[X_i^2] = t_i - t_{i-1}$,

$$\left\| \sum_i X_i^2 - t \right\|_2^2 = \mathbb{E}\left[\left(\sum_i \left(X_i^2 - (t_i - t_{i-1})\right)\right)^2\right]$$

and since for a centered Gaussian r.v. Y, $\mathbb{E}[Y^4] = 3\mathbb{E}[Y^2]^2$, this is equal to

$$2\sum_i (t_i - t_{i-1})^2 \leq 2t \sup_i |t_{i+1} - t_i|,$$

which completes the proof. □

Theorem 1.19. *If M is a continuous local martingale, there exists a unique increasing continuous process $\langle M, M\rangle$, vanishing at zero, such that $M^2 - \langle M, M\rangle$ is a continuous local martingale. Moreover, for every t and for any sequence $\{\Delta_n\}$ of subdivisions of $[0,t]$ such that $|\Delta_n| \to 0$, the r.v.'s*

$$\sup_{s \leq t} \left| T_s^{\Delta_n}(M) - \langle M, M\rangle_s \right|$$

converge to zero in probability.

Proof. See Theorem (1.8), Chapter IV, p. 124 in [4]. □

If M and N are two local martingale we get their "bracket product" by polarization:

$$\langle M, N\rangle_t = \frac{1}{2}\left(\langle M+N, M+N\rangle_t - \langle M, M\rangle_t - \langle N, N\rangle_t\right).$$

Definition 1.20. The process $\langle M, N\rangle$ is called the bracket of M and N, and the process $\langle M, M\rangle$ is called the increasing process associated with M or simply the increasing process of M.

A slightly generalized concept form martingale will ease our discussion in next section.

Definition 1.21. A process $X = (X_t)_{t\geq 0}$ is a continuous semimartingale if it can be written as $X_t = M + A$ where M is a local martingale, A is a finite variation process.

Proposition 1.22. *A continuous semimartingale $X = M + A$ has a finite quadratic variation and we have $\langle X, X\rangle = \langle M, M\rangle$.*

Proof. If Δ is a subdivision of $[0,t]$,

$$\left|\sum_i \left(M_{t_{i+1}} - M_{t_i}\right)\left(A_{t_{i+1}} - A_{t_i}\right)\right| \leq \left(\sup_i \left|M_{t_{i+1}} - M_{t_i}\right|\right) \mathrm{Var}_t(A)$$

where $\mathrm{Var}_t(A)$ is the variation of A on $[0,t]$, and this converges to zero when $|\Delta|$ tends to zero because of the continuity of M. Likewise

$$\lim_{|\Delta|\to 0} \sum_i \left(A_{t_{i+1}} - A_{t_i}\right)^2 = 0$$

□

1.3 Stochastic integral

When $\{\omega \in \Omega : X(t,\omega) = Y(t,\omega), \forall t \geq 0\}$ is measurable, which is usually true when some regularity properties hold for the trajectories of the two processes, we can say that X and Y are indistinguishable if

$$\mathbb{P}[X(t) = Y(t), \forall t \geq 0] = 1.$$

In this section, we introduce some basic techniques and notions which will be used throughout the sequel. Once and for all, we consider below, a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and we suppose that each $\mathcal{F}_t$ contains all the sets of $\mathbb{P}$-measure zero in $\mathcal{F}$. As a result, any limit (almost-sure, in the mean, etc.) of adapted processes is an adapted process; a process which is indistinguishable from an adapted process is adapted.

Let A be a continuous process with finite variation. One can clearly integrate appropriate functions with respect to the measure associated to $A(\omega)$ and thus obtain a "stochastic integral". More precisely, if X is progressively measurable and—for instance—bounded on every interval $[0,t]$ for a.e. ω, one can define for a.e. ω, the Stieltjes integral

$$(X \cdot A)_t(\omega) = \int_0^t X_s(\omega)\,\mathrm{d}A_s(\omega).$$

If ω is in the set where $A.(\omega)$ is not of finite variation or $X.(\omega)$ is not locally integrable with respect to $\mathrm{d}A(\omega)$, we put $(X \cdot A) = 0$. We then check that the process $X \cdot A$ thus defined is of finite variation. The hypothesis that X be progressively measurable is precisely made to ensure that $X \cdot A$ is adapted (this is a proposition from measure theory, see Corollary (3.3.3), Chapter 3, p.182 in [2] for more detail). It is the "stochastic integral" of X with respect to the process A of finite variation.

We will indulge in the usual confusion between processes and classes of indistinguishable processes in order to get norms and not merely semi-norms in the discussion below.

Definition 1.23. We denote by H^2 the set of L^2-bounded continuous martingales, i.e., the space of continuous $(\mathcal{F}_t, \mathbb{P})$-martingales M such that

$$\sup_t \mathbb{E}\left[M_t^2\right] < +\infty.$$

and H_0^2 the subset of elements of H^2 vanishing at zero.

Definition 1.24. If $M \in H^2$, we call $\mathscr{L}^2(M)$ the space of progressively measurable processes K such that

$$\|K\|_M^2 = \mathbb{E}\left[\int_0^\infty K_s^2 \,\mathrm{d}\langle M, M\rangle_s\right] < +\infty$$

As usual, $L^2(M)$ will denote the space of equivalence classes of elements of $\mathscr{L}^2(M)$; it is of course a Hilbert space for the norm $\|\cdot\|_M$.

Theorem 1.25. *Let $M \in H^2$; for each $K \in L^2(M)$, there is a unique element of H_0^2 denoted by $K \cdot M$, such that*

$$\langle K \cdot M, N\rangle = K \cdot \langle M, N\rangle.$$

Proof. See Theorem (2.2), Chapter IV, p.127 in [4]. □

Definition 1.26. The martingale $K \cdot M$ is called the stochastic integral (also called the Itô integral) of K with respect to M and is also denoted by

$$\int_0^{\cdot} K_s \,\mathrm{d}M_s.$$

The resulting process of Itô integral is a martingale.

Remark 1.27. Since the Brownian motion (one-dimensional Wiener process) stopped at a fixed time t is in H^2, if K is a process which satisfies

$$E\left[\int_0^t K_s^2 \,\mathrm{d}s\right] < +\infty, \qquad \text{for all } t,$$

we can define $\int_0^t K_s \, \mathrm{d}W_s$ for each t hence on the whole positive half-line and the resulting process is a martingale although not an element of H^2.

1.4 Itô formula

This section is fundamental. It is devoted to a "change of variables" formula for stochastic integrals which makes them easy to handle and thus leads to explicit computations. Another way of viewing this formula is to say that we are looking for functions which operate on the class of continuous semimartingales, that is, functions F such that $F(X_t)$ is a continuous semimartingale whatever the continuous semimartingale X is. However, to be fully prepared for it, we need to extend our result to semimartingale in previous section without proof here.

Definition 1.28. If M is a continuous local martingale, we call $L^2_{\text{loc}}\,(M)$ the space of classes of progressively measurable processes K for which there exists a sequence (T_n) of stopping times increasing to infinity and such that

$$\mathbb{E}\left[\int_0^{T_n} K_s^2 \, \mathrm{d}\langle M, M\rangle_s\right] < +\infty.$$

Observe that $L^2_{\text{loc}}\,(M)$ consists of all the progressive processes K such that

$$\int_0^t K_s^2 \, \mathrm{d}\langle M, M\rangle_s < \infty \text{ for every } t.$$

Proposition 1.29. *For any $K \in L^2_{\text{loc}}(M)$, there exists a unique continuous local martingale vanishing at 0 denoted $K \cdot M$ such that for any continuous local martingale N*

$$\langle K \cdot M, N\rangle = K \cdot \langle M, N\rangle.$$

Proof. See proposition (2.7), Chpater IV, p.120 in [4]. □

Definition 1.30. If K is locally bounded and $X = M + A$ is a continuous semimartingale, the stochastic integral of K with respect to X is the continuous semimartingale

$$K \cdot X = K \cdot A + K \cdot M$$

where $K \cdot M$ is the integral of Proposition 1.29 and $K \cdot A$ is the pathwise Stieltjes integral with respect to $\mathrm{d}A$. The semimartingale $K \cdot X$ is also written

$$\int_0^{\cdot} K_s \, \mathrm{d}X_s.$$

Theorem 1.31 (Itô formula)**.** *Let X be a semimartingale and $F : \mathbb{R} \to \mathbb{R}$ a function of class C^2, then*

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t F''(X_s)\,\mathrm{d}\langle X, X\rangle_s.$$

And if we consider d continuous semimartingales $X^1, \ldots, X^d$ and $F : \mathbb{R}^d \to \mathbb{R}$ of class C^2 then,

$$\begin{aligned} F\left(X_t^1, \ldots, X_t^d\right) =& F\left(X_0^1, \ldots, X_0^d\right) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}\left(X_s^1, \ldots, X_s^d\right)\mathrm{d}X_s^i \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}\left(X_s^1, \ldots, X_s^d\right)\mathrm{d}\left\langle X^i, X^j\right\rangle_s \end{aligned}$$

Proof. See Theorem (3.3), Chapter IV, p.147 in [2]. □

We often apply Itô formula in a special case $F(x) = x^2$.

Proposition 1.32 (Integration by parts)**.** *If X and Y are two continuous semimartingale, we have*

$$X_tY_t = X_0Y_0 + \int_0^t X_s\,\mathrm{d}Y_s + \int_0^t Y_s\,\mathrm{d}X_s + \langle X, Y\rangle_t$$

The term $\langle X, Y\rangle$ is zero if X or Y has finite variation.

We then give an application of Itô formula. Many continuous processes we care about in practice satisfies an equation of the form

$$X_t = x_0 + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}W_s,$$

or in a differential form,

$$\begin{cases} \mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t \\ X_0 = x_0 \end{cases}.$$

As we saw with Itô formula, there is a close link between probability theory and classic differential equations, and this type of stochastic differential equation (simply called SDE, we will define it formally later) above allows us to switch from one to the other. To illustrate our point, we will introduce some classic SDEs and show how to find their solutions. On the other hand, as soon as the equation becomes more complex, as in the deterministic case, it turns out that those simple methods are no longer accessible and then the question of existence and uniqueness of solutions arises.

Let's start with a simple SDE:

$$\begin{cases} \mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}W_t \\ X_0 = x_0 > 0 \end{cases}$$

where the constants μ and σ are in $\mathbb{R}$ and $]0, \infty[$ respectivly. To solve this SDE, we will use the Itô formula and look for a solution of the form $X_t = f(W_t, t)$. Then we get:

$$\mathrm{d}X_t = f_x(W_t, t)\,\mathrm{d}W_t + \left(\frac{1}{2} f_{xx}(W_t, t) + f_t(W_t, t)\right)\mathrm{d}t$$

where f_x et f_t are the first derivatives of f in respectively space and time and f_{xx} is the second derivative in space. By identifying the coefficients, we have:

$$\begin{cases} \mu f(x,t) = \frac{1}{2} f_{xx}(x,t) + f_t(x,t) \\ \sigma f(x,t) = f_x(x,t) \end{cases}$$

A solution of the second equation is of the form $f(x,t) = \exp(\sigma x + g(t))$, where g is an arbitrary function. So, by substituting it into the first equation, we find that g has to satisfy $g'(t) = \mu - \sigma^2/2$. As a result, we get a solution

$$X_t = x_0 \exp\left(\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right)$$

For the moment, we must admit that there may be other solutions to this SDE. We will see later that in fact it is the unique solution. Regarding this process, we note a strange phenomenon: since $W_t/t \to 0$ $a.s.$ when $t \to \infty$, we know that $X_t \to 0$ $a.s.$ in the case where $\sigma^2 > 2\mu$; but we also have $\mathbb{E}[X_t] = x_0 e^{\mu t}$. That is to say, X_t tend towards 0 $a.s.$ whereas on average it tends towards infinity at an exponential rate.

This SDE is a particular easy case of a more general class of SDE where we can get a theorem of existence and uniqueness of solution under some conditions that is close to Cauchy-Lipschitz condition in ODE.

1.5 Solution of SDE

To be strict in the sense of mathematics, in the section we give a formal description of SDE with coefficients which are non-random functions of the unknown process taken only at the last time. We fix a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, and consider an equivalent relation: two processes X and X' are equivalent if

$$\mathbb{P}\left[\int_0^{+\infty} \left|X(t) - X'(t)\right| \,\mathrm{d}t = 0\right] = 1.$$

We define $\mathcal{L}^p$ as the linear space of the (equivalence classes of) progressively measurable complex processes X such that

$$\mathbb{P}\left[\int_0^t |X(s)|^p \,\mathrm{d}s < +\infty\right] = 1, \quad \forall t \geq 0.$$

A process $\{X(t), t \geq t_0 \geq 0\}$ is called Itô process if it is a continuous, adapted process such that, for every $t \geq t_0$

$$X_i(t) = X_i(t_0) + \int_{t_0}^t F_i(s)\,\mathrm{d}s + \sum_{j=1}^d \int_{t_0}^t G_{ij}(s)\,\mathrm{d}W_j(s), \quad i = 1, \ldots, n.$$

with $X_i(t_0)$ being $\mathcal{F}_{t_0}$-measurable and $F_i \in \mathcal{L}^1, G_{ij} \in \mathcal{L}^2$. W_j here is the jth component of a d-dimensional Wiener process. It is usual to say that X has initial value $X_i(t_0)$ and it admits the stochastic differential

$$\mathrm{d}X_i(t) = F_i(t)\,\mathrm{d}t + \sum_{j=1}^d G_{ij}(t)\,\mathrm{d}W_j(t).$$

Note that if we apply Itô formula to $f(X_t, t)$ when f is twice differentiable, we then get another Itô process:

$$\begin{aligned} f(X(t), t) = & f(X(t_0), t_0) + \int_{t_0}^t \Bigg[f_t(X(s), s) + \sum_i f_i(X(s), s) F_i(s) \\ & + \frac{1}{2} \sum_{ikj} f_{ik}(X(s), s) G_{ij}(s) G_{kj}(s) \Bigg] \mathrm{d}s \\ & + \sum_{ij} \int_{t_0}^t f_i(X(s), s) G_{ij}(s) \mathrm{d}W_j(s) \end{aligned}$$

where we set $f_t(x,t) := \frac{\partial f(x,t)}{\partial t}, \quad f_i(x,t) := \frac{\partial f(x,t)}{\partial x_i}, \quad f_{ik}(x,t) := \frac{\partial^2 f(x,t)}{\partial x_i \partial x_k}$.

Hypothesis 1.33. *Let b and $\sigma_j, j = 1, \ldots, d$, be (Borel) measurable deterministic functions from $\mathbb{C}^n \times [t_0, T]$ to $\mathcal{H} := \mathbb{C}^n$.*

Deterministic here means independent of $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the SDE

$$\mathrm{d}X(t) = b(X(t), t)\,\mathrm{d}t + \sum_{j=1}^d \sigma_j(X(t), t)\,\mathrm{d}W_j(t) \tag{1.1}$$

for processes X with values in $\mathbb{C}^n$.

A solution of (1.1) with initial condition $X(t_0) = \eta$ is an Itô process satisfying (*a.s.*, $\forall t \geq t_0$)

$$X(t) = \eta + \int_{t_0}^{t} b(X(s), s)\,\mathrm{d}s + \sum_{j=1}^{d} \int_{t_0}^{t} \sigma_j(X(s), s)\,\mathrm{d}W_j(s) \tag{1.2}$$

Definition 1.34. The SDE (1.1) admits strong solutions if, for any choice of a stochastic basis satisfying usual conditions with a Wiener process W and for every $x_0 \in \mathcal{H}$, there exists a continuous adapted process X such that Eq. (1.2) holds with $\eta = x_0$.

Definition 1.35. A weak solution of the SDE (1.1) with an initial condition with law μ is a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions, with a Wiener process W, an $\mathcal{H}$-valued $\mathcal{F}_{t_0}$-measurable random variable $\eta \sim \mu$ and an adapted, continuous process X such that for every $t \geq t_0$ Eq. (1.2) holds.

Definition 1.36. The solution of the SDE (1.1) is unique in law if, taken any two solutions $(\Omega, \mathcal{F}, (\mathcal{F}_t), P), W, \eta, X$ and $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P'), W', \eta', X'$, with $\eta \sim \eta'$ then the processes X and X' have the same law.

Definition 1.37. The solution of the SDE (1.1) is pathwise unique if taken any two solutions $(\Omega, \mathcal{F}, (\mathcal{F}_t), P), W, \eta, X$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t), P), W, \eta, X'$, then the processes X and X' are indistinguishable.

1.6 Lipschitz Condition of SDE

The norm $\|A\|$ for a matrix A will not be specified but must be fixed, which means these propositions stay valid for any particular matrix norm. We present a set of hypotheses which imply existence and uniqueness of the solution of our SDE (1.1).

Hypothesis 1.38 (Global Lipschitz condition)**.** *There exists a constant $L(T) > 0$ such that*

$$\|b(x,t) - b(y,t)\|^2 + \sum_j \|\sigma_j(x,t) - \sigma_j(y,t)\|^2 \leq L(T)\|x - y\|^2$$

for all $x, y \in \mathbb{C}^n$ and $t \in [t_0, T]$

Hypothesis 1.39 (Linear growth condition)**.** *There exists a constant $M(T) > 0$ such that*

$$\|b(x,t)\| + \left(\sum_j \|\sigma_j(x,t)\|^2\right)^{1/2} \leq M(T)(1 + \|x\|)$$

for all $x \in \mathbb{C}^n$ and $t \in [t_0, T]$

Theorem 1.40 (existence-and-uniqueness theorem)**.** *Under Hypotheses 1.33, 1.38, 1.39 the SDE (1.1) admits strong solutions in $[t_0, T]$. Pathwise uniqueness and uniqueness in* *law hold. And the SDE* (1.1) *with initial condition* $\eta \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathcal{H})$ *has a pathwise unique solution* X *in* $[t_0, T]$*. The solution satisfies*

$$\int_{t_0}^{T} \mathbb{E}\left[\|X(t)\|^2\right] \mathrm{d}t < +\infty.$$

If Hypotheses 1.33, 1.38, 1.39 hold for every $T > 0$*, then the SDE* (1.1) *admits a unique strong solution in* $[t_0, \infty)$*.*

Proof. Our conditions are stronger than what we need, see Theorem 2.9, Chapter 5, p. 289 in [3]. □

When the assumptions of the existence-and-uniqueness theorem hold for every $T > 0$, the unique strong solution in $[t_0, \infty)$ is called a global solution.

1.7 Doléans equation and Girsanov Theorem

Doléans equation is another class of SDE where the coefficient is no longer deterministic. Again, let W be a d-dimensional Wiener process defined in a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions. Let us take some stochastic processes $F \in \mathcal{L}^1, G_j \in \mathcal{L}^2, j = 1, \ldots, d$, and let us introduce the complex Itô process

$$X(t) := \sum_{j=1}^{d} \int_0^t G_j(s)\,\mathrm{d}W_j(s) + \int_0^t F(s)\,\mathrm{d}s \tag{1.3}$$

Then, we consider the exponential of X times a generic constant:

$$Z(t) := z_0 \exp\{X(t)\}, \quad z_0 \in \mathbb{C} \tag{1.4}$$

The process Z is an Itô process, and by Itô formula, we get

$$Z(t) = z_0 + \sum_{j=1}^{d} \int_0^t Z(s) G_j(s)\,\mathrm{d}W_j(s) + \int_0^t Z(s)\left[F(s) + \frac{1}{2}\sum_{j=1}^{d} G_j(s)^2\right] \mathrm{d}s, \tag{1.5}$$

that is

$$\begin{cases} \mathrm{d}Z(t) = \sum_{j=1}^{d} Z(t) G_j(t)\,\mathrm{d}W_j(t) + Z(t)\left[F(t) + \frac{1}{2}\sum_{j=1}^{d} G_j(t)^2\right] \mathrm{d}t \\ Z(0) = z_0 \end{cases} \tag{1.6}$$

The integrals in Eq. (1.3) are well-defined and, so, $\mathbb{P}[|X(t)| < +\infty] = 1$. For $z_0 \neq 0$, this implies $\mathbb{P}[Z(t) = 0] = 0$ and, so, $Z(t)^{-1}$ is a *bona fide* random variable. Obviously, $Z(t)^{-1} = \exp\{-X(t)\}/z_0$ and by Itô formula we get

$$\begin{aligned} Z(t)^{-1} =& \frac{1}{z_0} - \sum_{j=1}^{d} \int_0^t Z(s)^{-1} G_j(s)\, \mathrm{d}W_j(s) \\ &+ \int_0^t Z(s)^{-1} \left[-F(s) + \frac{1}{2} \sum_{j=1}^{d} G_j(s)^2 \right] \mathrm{d}s. \end{aligned} \tag{1.7}$$

Proposition 1.41. *For $F \in \mathcal{L}^1, G_j \in \mathcal{L}^2, j = 1, \ldots, d$, the process Z, defined by Eqs.(1.3), (1.4), is the pathwise unique solution of the Doléans equation (1.6).*

Proof. Let Y be another solution of (1.6). This means that Y is a continuous, adapted process, that $YG_j \in \mathcal{L}^2, Y\left[F + \frac{1}{2}\sum_{j=1}^{d} G_j^2\right] \in \mathcal{L}^1$ and that (1.5) holds with Z replaced by Y. Then $Y(0)\exp\{-X(0)\} = z_0$, and Itô formula and Eq. (1.7) imply $\mathrm{d}(Y(t)\exp\{-X(t)\}) = 0$. Being continuous processes we obtain that $Y(t)\exp\{-X(t)\} = z_0$ for every $t \geq 0$ *a.s.*, so that Y and Z are indistinguishable. □

We state Girsanov theorem without proof, see Chapter VIII, p.326 in [4] for detailed discussion. Let $(\mathcal{F}_t), t \geq 0$, be a right-continuous filtration with terminal σ-field $\mathcal{F}_\infty$ and $\mathbb{P}$ and $\mathbb{Q}$ two probability measures on $\mathcal{F}_\infty$. We assume that for each $t \geq 0$, the restriction of $\mathbb{Q}$ to $\mathcal{F}_t$ is absolutely continuous with respect to the restriction of $\mathbb{P}$ to $\mathcal{F}_t$, which will be denoted by $\mathbb{Q} \lhd \mathbb{P}$.

Proposition 1.42. *If D is a strictly positive continuous local martingale, there exists a unique continuous local martingale L such that*

$$D_t = \exp\left\{ L_t - \frac{1}{2}\langle L, L\rangle_t \right\} = \mathscr{E}(L)_t;$$

L is given by the formula

$$L_t = \log D_0 + \int_0^t D_s^{-1} dD_s.$$

If $\mathbb{P}$ and $\mathbb{Q}$ are equivalent on each $\mathcal{F}_t$, we then have $\mathbb{Q} = \mathscr{E}(L)_t \cdot \mathbb{P}$ on $\mathcal{F}_t$ for every t, which we write simply as $\mathbb{Q} = \mathscr{E}(L) \cdot \mathbb{P}$.

Theorem 1.43. *If $\mathbb{Q} = \mathscr{E}(L) \cdot \mathbb{P}$ and M is a continuous $\mathbb{P}$-local martingale, then*

$$\widetilde{M} = M - D^{-1} \cdot \langle M, D\rangle = M - \langle M, L\rangle$$

is a continuous $\mathbb{Q}$-local martingale. Moreover, $\mathbb{P} = \mathscr{E}(-\widetilde{L}) \cdot \mathbb{Q}$

In particular, if we consider the Brownian motion (one-dimensional Wiener process, simply written as BM), we should have following result.

Theorem 1.44. *If* $\mathbb{Q} \lhd \mathbb{P}$ *and if* B *is a* $(\mathcal{F}_t, \mathbb{P})$*-BM, then* $\widetilde{B} = B - \langle B, L\rangle$ *is a* $(\mathcal{F}_t, \mathbb{Q})$*-BM.*

Chapter 2

Quantum Mechanics

We discuss two statistic formulations of Schrödinger equation in quantum mechanics, linear and non-linear form, separately in the first section and the third section. The former, as we may deduce from the word "linear", behaves well in theory but appears indirect in intuitive explanation; the non-linear one, called stochastic master equation in physical literature, does the opposite. In the second section, we apply results from previous chapter to study the existence and uniqueness of solution to the linear stochastic master equation.

2.1 The Linear Stochastic Master Equation

We identify linear operators in finite dimension with their matrix representation. The trace of an operator A is $\mathrm{Tr}\{A\} = \sum_i A_{ii}$, which does not depend on the chosen basis. Recall that, for every $a, b \in \mathbb{C}$,

$$\mathrm{Tr}\{aA + bB\} = a\,\mathrm{Tr}\{A\} + b\,\mathrm{Tr}\{B\}, \quad \mathrm{Tr}\{AB\} = \mathrm{Tr}\{BA\}$$

Let us denote by $\mathcal{S}(\mathcal{H})$ the set of all statistical operators on a finite vector space $\mathcal{H}$:

$$\mathcal{S}(\mathcal{H}) := \{ \text{ all operators } \rho \text{ on } \mathcal{H} \text{ such that: } \rho^* = \rho, \rho \geq 0, \mathrm{Tr}\{\rho\} = 1 \} .$$

Assumption 2.1. *The initial state is a statistical operator $\rho_0 \in \mathcal{S}(\mathcal{H})$, where $\mathcal{H} \equiv \mathbb{C}^n$.*

The initial state depends on the way the system has been experimentally prepared and it determines the probability distributions of every measurement performed on the system.. To introduce the linear stochastic master equation, we give the restrictions on its coefficients in the next assumption. We define the commutator $[A,B] := AB - BA$ and anti-commutator $\{A,B\} := AB + BA$. And we use the operator norm,

$$\|A\| \equiv \|A\|_\infty := \sup_{\psi\in\mathcal{H}:\|\psi\|=1} \|A\psi\|. \tag{2.1}$$

Assumption 2.2. *The process W is a continuous m-dimensional Wiener process in a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ satisfying usual conditions and $\mathcal{F} = \mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t$; W has increments independent of the past. The maps $\mathcal{R}_j(t), \mathcal{L}(t)$ are linear operators over the space M_n of $n\times n$ complex matrices τ with the structure*

$$\begin{array}{c} \mathcal{R}_j(t)[\tau] = R_j(t)\tau + \tau R_j(t)^* \\ \mathcal{L}(t) = \mathcal{L}_0(t) + \mathcal{L}_1(t) \end{array},$$

where

$$\mathcal{L}_1(t)[\tau] = \sum_{j=1}^{m} \left(R_j(t)\tau R_j(t)^* - \frac{1}{2}\left\{R_j(t)^* R_j(t), \tau\right\}\right)$$

$$\mathcal{L}_0(t)[\tau] = -\mathrm{i}[H(t),\tau] + \sum_{j=m+1}^{d} \left(R_j(t)\tau R_j(t)^* - \frac{1}{2}\left\{R_j(t)^* R_j(t), \tau\right\}\right)$$

The coefficients $R_j(t), H(t)$ are (non-random) linear operators on $\mathcal{H} \equiv \mathbb{C}^n$ and $H(t) = H(t)^*$. The functions $t \mapsto H(t)$ and $t \mapsto R_j(t)$ are measurable and such that $\forall T \in (0,+\infty)$,

$$\sup_{t\in[0,T]} \|H(t)\| < +\infty, \qquad \sup_{t\in[0,T]} \left\| \sum_{j=1}^{d} R_j(t)^* R_j(t) \right\| < +\infty. \tag{2.2}$$

The linear stochastic master equation is defined as, for an operator-valued process σ :

$$\begin{cases} \mathrm{d}\sigma(t) = \mathcal{L}(t)[\sigma(t)]\,\mathrm{d}t + \sum_{j=1}^m \mathcal{R}_j(t)[\sigma(t)]\,\mathrm{d}W_j(t) \\ \sigma(0) = \rho_0 \in \mathcal{S}(\mathcal{H}) \end{cases}, \tag{2.3}$$

where m and d are two positive integers and $m \leq d$.

They are the only two assumptions needed in our discussion and here we collect their heuristic interpretation in theory of continuous measurements.

- ρ_0 is the initial state of the quantum system;
- $(\Omega, \mathcal{F})$ is the measurable space of the possible outcomes of our experiment;
- $\mathcal{F}_t$ is the collection of events verifiable already at time t;
- the m stochastic processes $W_j(t)$ are the output of the continuous measurement and their derivatives $\mathrm{d}W_j(t)$ can be interpreted as instantaneous imprecise measurements of the quantum observables $R_j(t) + R_j(t)^*$ performed at time t;

- the operator $H(t)$ has a role of effective Hamiltonian of the system and that the operators $R_j(t)$ with indexes $j = m+1, \dots, d$ characterise the unobserved channels. On the other side we say that the operators $R_j(t)$ with indexes $j = 1, \dots, m$, appearing as coefficients in the diffusive part of SDE (2.3) and in $\mathcal{L}_1(t)$, characterise the observed channels. If $m < d$, that is if the Liouvillian $\mathcal{L}_0(t)$ is not simply Hamiltonian, we say that the measurement is incomplete.

- the structure of maps $\mathcal{R}_j(t), \mathcal{L}(t)$ are determined by the physical probabilities that involves martingale properties.

We consider also the fundamental solution $\mathcal{A}(t,s)$ of the linear SDE (2.3), defined by

$$\begin{cases} \mathrm{d}\mathcal{A}(t,s) = \mathcal{L}(t) \circ \mathcal{A}(t,s)\,\mathrm{d}t + \sum_{j=1}^{m} \mathcal{R}_j(t) \circ \mathcal{A}(t,s)\,\mathrm{d}W_j(t) \\ \mathcal{A}(s,s) = \mathrm{Id}_n \end{cases} \tag{2.4}$$

We call $\mathcal{A}(t,s)$ the stochastic evolution map or, borrowing a terminology used in theoretical physics, the propagator associated to the linear SDE.

2.2 Existence and Uniqueness of Solution to Linear Statistical SDE

We introduce the natural two-times filtrations of W :

$$\mathcal{G}_t^s := \sigma\left\{W_j(r) - W_j(s), r \in [s,t], j = 1, \dots, m\right\}$$

$$\mathcal{G} := \bigvee_{t \geq 0} \mathcal{G}_t^0, \quad \mathcal{N}_{\mathcal{G}} := \{A \in \mathcal{G} : \mathbb{Q}(A) = 0\}$$

$$\bar{\mathcal{G}}_t^s := \mathcal{G}_t^s \vee \mathcal{N}_{\mathcal{G}}$$

Let us note that

$$\mathcal{G} \subset \mathcal{F}, \quad \mathcal{N}_{\mathcal{G}} \subset \mathcal{N} \equiv \{A \in \mathcal{F} : \mathbb{Q}(A) = 0\}, \quad \mathcal{G}_t^s \subset \bar{\mathcal{G}}_t^s \subset \mathcal{F}_t$$

Moreover, $\left(\Omega, \mathcal{G}, \left(\bar{\mathcal{G}}_t^0\right), \mathbb{Q}\right)$ is a stochastic basis satisfying usual conditions. $\bar{\mathcal{G}}_t^0$ is the collection of events verifiable already at time t which effectively regard the continuous measurement.

Theorem 2.3. *Under Assumptions 2.2, the linear stochastic master equation (2.3) admits continuous strong solutions in $[0, +\infty)$. Pathwise uniqueness and uniqueness in law hold. The solution $\sigma(t)$ of (2.3) with initial condition $\sigma(0) = \rho_0 \in \mathcal{S}(\mathcal{H})$, is non-negative.*

Moreover, $p(t) := \mathrm{Tr}\{\sigma(t)\}$ is a mean one $\mathbb{Q}$-martingale, it is a.s. strictly positive and it can be written as

$$p(t) = \mathrm{Tr}\{\sigma(t)\} = \exp\left\{\sum_{j=1}^{m}\left[\int_0^t v_j(s)\,\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t v_j(s)^2\,\mathrm{d}s\right]\right\} \tag{2.5}$$

where

$$v_j(t) := \mathrm{Tr}\left\{\left(R_j(t) + R_j(t)^*\right)\rho(t)\right\} = 2\,\mathrm{Re}\,\mathrm{Tr}\left\{R_j(t)\rho(t)\right\}, \tag{2.6}$$

$$\rho(t) := p(t)^{-1}\sigma(t) \tag{2.7}$$

The linear SDE (2.4) admits strong solutions in $(s, +\infty)$, for every $s \geq 0$. Pathwise uniqueness and uniqueness in law hold. $\mathcal{A}(t,s)$ is $\mathbb{Q}$-independent of $\mathcal{F}_s, \bar{\mathcal{G}}_t^s$ measurable, positive, i.e., mapping positive matrices to positive matrices, and continuous in t. Moreover, for $0 \leq r \leq s \leq t$ one has a.s.

$$\mathcal{A}(t,s) \circ \mathcal{A}(s,r) = \mathcal{A}(t,r), \quad \sigma(t) = \mathcal{A}(t,0)\left[\rho_0\right] \tag{2.8}$$

The master equation

$$\mathcal{T}(t,s) = \mathrm{Id}_n + \int_s^t \mathcal{L}(r) \circ \mathcal{T}(r,s)\,\mathrm{d}r \tag{2.9}$$

admits a unique solution in $[s, +\infty)$, for every $s \geq 0$. Moreover, the solution admits the representation

$$\mathcal{T}(t,s) = \mathbb{E}_{\mathbb{Q}}[\mathcal{A}(t,s)] \tag{2.10}$$

it is continuous in t, positive, trace preserving, and it satisfies the composition law :

$$\mathcal{T}(t,r) = \mathcal{T}(t,s) \circ \mathcal{T}(s,r), \quad 0 \leq r \leq s \leq t. \tag{2.11}$$

Proof. Equation (2.3) is for an $(n \times n)$-dimensional process and (2.4) for an $(n^2 \times n^2)$-dimensional one; in both cases we have finite dimensional processes. The bounds (2.2) and the linearity give that the global Lipschitz condition 1.38 and the linear growth condition 1.39 hold. Then, as the measurability condition 1.33 trivially holds, Theorem 1.40 gives the existence of strong solutions and the uniqueness statements for both SDEs.

By completeness, let us check in detail Hypotheses 1.38 and 1.39 for $t \in [0, T]$. We make use of following two norms, the Hilbert-Schmidt norm:

$$\|A\|_2 := \sqrt{\mathrm{Tr}\left\{A^* A\right\}} = \sqrt{\sum_{ij} |A_{ij}|^2}$$

and the trace norm:

$$\|A\|_1 := \mathrm{Tr}\{\sqrt{A^* A}\}.$$

First of all, let us note that

$$\|\tau^*\tau\|_1 = \|\tau\tau^*\|_1 = \|\tau\|_2^2,$$

which follows from the definitions of the two norms and from the positivity of $\tau^*\tau$ and $\tau\tau^*$. Moreover, for any matrix A we have, recalling that $\|A\|$ still stands for operator norm (2.1),

$$\|A\tau\|_2^2 = \operatorname{Tr}\{\tau^* A^* A\tau\} = \operatorname{Tr}\{A^* A\tau\tau^*\} \leq \|A^* A\| \, \|\tau\tau^*\|_1 = \|A\|^2 \|\tau\|_2^2$$

so that

$$\begin{aligned}\|A\tau A^*\|_2^2 &= \operatorname{Tr}\{A\tau^* A^* A\tau A^*\} = \operatorname{Tr}\{A^* A\tau^* A^* A\tau\} \\ &\leq \|A^* A\tau^*\|_2 \, \|A^* A\tau\|_2 \leq \|A^* A\| \, \|\tau^*\|_2 \, \|A^* A\| \, \|\tau\|_2 = \|A^* A\|^2 \, \|\tau\|_2^2\end{aligned}$$

We also set

$$\ell_T := \max\left(\sup_{0\leq t\leq T} \|H(t)\|, \sup_{0\leq t\leq T} \left\| \sum_{j=1}^d R_j(t)^* R_j(t) \right\| \right);$$

by (2.2), $\ell_T < +\infty$. Since $\sum_j R_j(t)^* R_j(t)$ is a sum of positive operators, we have also

$$\|R_j(t)\|^2 = \|R_j(t)^* R_j(t)\| \leq \ell_T.$$

In the case of (2.3) the relevant norm, needed in Hypotheses 1.38 and 1.39, is the Hilbert-Schmidt norm. We have

$$\begin{aligned}\|\mathcal{L}(t)[\tau]\|_2 &\leq 2\|H(t)\tau\|_2 + \sum_{j=1}^d \|R_j(t)\tau R_j(t)^*\|_2 + \left\| \sum_{j=1}^d R_j(t)^* R_j(t)\tau \right\|_2 \\ &\leq 2\|H(t)\|\|\tau\|_2 + \sum_{j=1}^d \|R_j(t)^* R_j(t)\| \, \|\tau\|_2 + \left\| \sum_{j=1}^d R_j(t)^* R_j(t) \right\| \|\tau\|_2 \\ &\leq (3+d)\ell_T \|\tau\|_2,\end{aligned}$$

$$\sum_{j=1}^m \|\mathcal{R}_j(t)[\tau]\|_2^2 \leq 2\sum_{j=1}^m \left(\|R_j(t)\tau\|_2^2 + \|\tau R_j(t)^*\|_2^2 \right) \leq 4\sum_{j=1}^m \|R_j(t)\|^2 \, \|\tau\|_2^2 \leq 4m\ell_T \|\tau\|_2^2.$$

These two estimates imply both Hypothesis 1.38 and Hypothesis 1.39.

The proof of existence and uniqueness of the solution of SDE (2.4) is completely similar and it is based on the estimates

$$\sum_{k,l=1}^n \|\mathcal{L}(t) \circ \mathcal{A}(t;s)[|k\rangle\langle l|]\|_2^2 \leq (3+d)^2 \ell_T^2 \sum_{k,l=1}^n \|\mathcal{A}(t;s)[|k\rangle\langle l|]\|_2^2,$$

$$\sum_{k,l=1}^{n}\sum_{j=1}^{m}\|\mathcal{R}_j(t)\circ\mathcal{A}(t;s)[|k\rangle\langle l|]\|_2^2 \leq 4m\ell_T\sum_{k,l=1}^{n}\|\mathcal{A}(t;s)[|k\rangle\langle l|]\|_2^2,$$

where $\{|k\rangle\}_{k=1}^{n}$ is a basis in $\mathcal{H}$; here we use Dirac Notations to represent a vector: $\langle\psi| = (\psi_1,\psi_2,\dots,\psi_n)$ is row complex vector and $|\psi\rangle$ means its conjugated transpose.

The continuity in t of $\sigma(t)$ and $\mathcal{A}(t,s)$ comes from the fact that we are working in a stochastic basis in usual hypotheses and it is included in Definition 1.34 of strong solution. Because of the existence of strong solutions and pathwise uniqueness, the random variable $\mathcal{A}(t,s)$ is $\bar{\mathcal{G}}_t^s$-measurable; then, the statement about the $\mathbb{Q}$-independence of $\mathcal{F}_s$ follows from the independent-increment property of the Wiener process. Moreover, the two sides of the composition law in (2.8) satisfy the same SDE (2.4) for $t \leq s$ and so they are equal by the uniqueness statement of Theorem 1.40. Analogously, $\sigma(t)$ and $\mathcal{A}(t,0)\,[\rho_0]$ are $a.s.$ equal because they satisfy the same SDE (2.3) with the same initial condition.

To prove that $\mathcal{A}(t,s)$ is positive we need to introduce another SDE:

$$\begin{cases} \mathrm{d}A_t^s = K(t)A_t^s\,\mathrm{d}t + \sum_{j=1}^{d} R_j(t)A_t^s\,\mathrm{d}W_j(t) \\ A_s^s = \mathbb{1} \end{cases} \tag{2.12}$$

Once the operators $H(t)$ and $R_j(t)$ appearing in Assumption 2.2 have been fixed and the stochastic evolution operator A_t^s solution of the SDE (2.12) with $K(t) = -\mathrm{i}\,H(t) - \frac{1}{2}\sum_{j=1}^{d} R_j(t)^* R_j(t)$ has been constructed, one can check that the map $\mathbb{E}_{\mathbb{Q}}\left[A_t^s \bullet A_t^{s*}|\bar{\mathcal{G}}_t^s\right]$ satisfies the same SDE as $\mathcal{A}(t,s)$. Since a conditional expectation is a positive map and the same is true for a map of the type $\rho \mapsto A\rho A^*$, the map $\mathbb{E}_{\mathbb{Q}}\left[A_t^s \bullet A_t^{s*}|\bar{\mathcal{G}}_t^s\right]$ is positive. Therefore, by the uniqueness in law of the solution of the SDE (2.4) $\mathcal{A}(t,s)$ is positive, too.

We can see the master equation (2.9) as a particular case of SDE; by the properties of the coefficients and Theorem 1.40, for every $s \geq 0$ the solution is pathwise unique. Moreover, by Theorem 1.40, the mean of $\mathcal{A}(t,s)$ exists and the stochastic integral in (2.4) has mean zero, so that $\mathbb{E}_{\mathbb{Q}}[\mathcal{A}(t,s)]$ is well defined and satisfies (2.9). For what concerns the composition law, the two sides of (2.11) satisfy the same equation with the same initial condition; they are equal by uniqueness of the solution. The continuity in t follows from the integral representation (2.10), the positivity from the same property of $\mathcal{A}(t,s)$ and the trace preserving property from the structure of any Liouville operator, which guarantees $\mathrm{Tr}\{\mathcal{L}(r)[\tau]\} = 0$ for every operator τ.

The positivity of $\mathcal{A}(t,0)$ and $\rho \geq 0$, imply $\sigma(t) \geq 0$. By taking the trace of the linear stochastic master equation (2.3) we get

$$\mathrm{Tr}\{\sigma(t)\} = 1 + \sum_{j=1}^{m} \int_0^t 2\,\mathrm{Re}\left(\mathrm{Tr}\left\{R_j(s)\sigma(s)\right\}\right) \mathrm{d}W_j(s) \tag{2.13}$$

By the bound (2.2) and the estimate of Theorem 1.40 for the process σ, we have that the integrand in the equation above is in the class $\mathcal{M}^2$; therefore, the stochastic integral is a $\mathbb{Q}$-martingale by Remark 1.27. Let $\rho_\star$ be a fixed statistical operator and let us define

$$\rho(t) = \begin{cases} (\mathrm{Tr}\{\sigma(t)\})^{-1}\sigma(t), & \text{if } \mathrm{Tr}\{\sigma(t)\} > 0 \\ \rho_\star, & \text{if } \mathrm{Tr}\{\sigma(t)\} = 0 \end{cases}$$

Then, (2.13) can be written as

$$\mathrm{Tr}\{\sigma(t)\} = 1 + \sum_{j=1}^{m} \int_0^t \mathrm{Tr}\{\sigma(s)\} v_j(s)\, \mathrm{d}W_j(s)$$

where v_j is given by (2.6). The solution of this Doléans equation (1.6) is unique and it is given by (2.5). Being of exponential form, it is strictly positive with probability one. $\square$

2.3 The Stochastic Master Equation

Starting from the linear stochastic master equation (2.3) we introduce the physical probabilities, the *a posteriori* states and the *a priori* states. Let us recall that the initial state at time zero is the statistical operator $\rho_0 \in \mathcal{S}(\mathcal{H})$.

We define the adjoint of a linear map $\mathcal{O} : (M_n, \|\bullet\|_1) \to (M_n, \|\bullet\|_1)$ as the linear map $\mathcal{O}^* : (M_n, \|\bullet\|_\infty) \to (M_n, \|\bullet\|_\infty)$:

$$\mathrm{Tr}\left\{\mathcal{O}^*[A]B\right\} = \mathrm{Tr}\{A\mathcal{O}[B]\}, \quad \forall A, B \in M_n$$

A measurement on a quantum system can produce different results with some probability distribution depending on the state ρ. An event regarding such a measurement, which can occur or cannot, is represented by an effect E: a self-adjoint operator E such that $0 \leq E \leq \mathbb{1}$. We denote by $[0, \mathbb{1}]$ the set of all effects. Obviously, $E \leq \mathbb{1}$ means $\mathbb{1} - E \geq 0$.

Definition 2.4. We define the quantities

$$E_t^s(G) := \int_G \mathcal{A}(t,s;\omega)^*[\mathbb{1}]\mathbb{Q}(\mathrm{d}\omega) \equiv \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t,s)^*[\mathbb{1}]\right], \quad G \in \bar{\mathcal{G}}_t^s \tag{2.14}$$

$$\mathbb{P}^t_{\rho_0}(G) := \operatorname{Tr}\left\{E^0_t(G)\rho_0\right\} = \mathbb{E}_{\mathbb{Q}}\left[1_G \operatorname{Tr}\{\sigma(t)\}\right], \quad G \in \bar{\mathcal{G}}^0_t \tag{2.15}$$

Definition 2.5. Let $(\Omega, \mathcal{F})$ be a measurable space; a positive operator-valued measure (POM) is a map E from $\mathcal{F}$ into the set of the effects such that it is normalised and σ-additive, i.e., $E(\Omega) = \mathbb{1}$ and for any sequence $F_1, F_2, \dots$ of incompatible events (disjoint sets) one has $E\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} E\left(F_k\right)$.

Proposition 2.6. *E^s_t is a positive operator-valued measure on the value space $\left(\Omega, \bar{\mathcal{G}}^s_t\right)$ and $\mathbb{P}^t_{\rho 0}$ is a probability measure on the value space $\left(\Omega, \bar{\mathcal{G}}^0_t\right)$. The family of probability measures $\left\{\mathbb{P}^t_{\rho_0}, t > 0\right\}$ is consistent, i.e., $\mathbb{P}^t_{\rho_0}(G) = \mathbb{P}^s_{\rho_0}(G)$ for any $G \in \bar{\mathcal{G}}^0_s$ with $0 < s < t$. Analogously, we have the consistency of the POMs:*

$$0 \le s < t < T, \quad G \in \bar{\mathcal{G}}^s_t \quad \Rightarrow \quad E^s_t(G) = E^s_T(G)$$

Let T be an arbitrary positive time. Under the probability $\mathbb{P}^T_{\rho_0}$, the processes

$$\widehat{W}_j(t) := W_j(t) - \int_0^t v_j(s)\,\mathrm{d}s, \quad j = 1, \dots, m, \quad t \in [0, T] \tag{2.16}$$

are independent, $\left(\bar{\mathcal{G}}^0_t\right)$-adapted, standard Wiener processes.

Proof. By the properties of $\mathcal{A}(t,s)$ we have

$$0 \le \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t,s)^*[\mathbb{1}]\right] \le \mathbb{E}_{\mathbb{Q}}\left[\mathcal{A}(t,s)^*[\mathbb{1}]\right] = \mathcal{T}(t,s)^*[\mathbb{1}] = \mathbb{1}\ .$$

Then, from the Definition (2.14) one can check that all the properties characterising a POM hold. The consistency of the POMs follows from the composition property of the propagator $\mathcal{A}$, the independence of $1_G\mathcal{A}(t,s)^*$ and $\mathcal{A}(T,t)^*$ and $\mathcal{T}(T,t)^*[\mathbb{1}] = \mathbb{1}$:

$$\begin{aligned} E^s_T(G) &= \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(T,s)^*[\mathbb{1}]\right] = \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t,s)^* \circ \mathcal{A}(T,t)^*[\mathbb{1}]\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t,s)^* \circ \mathcal{T}(T,t)^*[\mathbb{1}]\right] = \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t,s)^*[\mathbb{1}]\right] = E^s_t(G) \end{aligned}$$

E^0_t being a POM and ρ_0 a state, Definition (2.15) defines a probability measure. Consistency follows from the fact that $\operatorname{Tr}\{\sigma(t)\}$ is a martingale or from the consistency of the POMs. The statement on $\widehat{W}(t)$ is from corollary 1.44 of Girsanov theorem where $M = W_j$,

$$L = \sum_{j=1}^m \left[\int_0^t v_j(s)\,\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t v_j(s)^2\,\mathrm{d}s\right]$$

and we have

$$\begin{aligned}\langle W_j, L\rangle &= \langle W_j, \sum_{j=1}^{m}\left[\int_0^t v_j(s)\,\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t v_j(s)^2\,\mathrm{d}s\right]\rangle \\ &= \langle W_j, \sum_{j=1}^{m}\left[\int_0^t v_j(s)\,\mathrm{d}W_j(s)\right]\rangle = \int_0^t v_j(s)\,\mathrm{d}s.\end{aligned}$$

□

The observables of the theory are represented by the POMs E_t^0 and the pre-measurement state by ρ_0. Then, the physical probabilities are defined by (2.15) and their structure in terms of a POM and a state guarantees that the usual axioms of quantum mechanics are not violated. Moreover, we can write

$$\mathbb{P}_{\rho_0}^t(\,\mathrm{d}\omega) = \mathrm{Tr}\{\sigma(t,\omega)\}\mathbb{Q}(\,\mathrm{d}\omega)|_{\bar{\mathcal{G}}_t^0}$$

The value space of the POM E_t^0 is $\left(\Omega, \bar{\mathcal{G}}_t^0\right)$, but $\bar{\mathcal{G}}_t^0$ is generated by W and this allows to identify the m-dimensional process W with the output. The output of the measurement has to be considered under the physical probability $\mathbb{P}_{\rho 0}^T$.

The random statistical operator $\rho(t)$ defined in (2.7) can be consistently interpreted as the state of the system at time t conditional on the output observed up to time t : for every $0 \le s \le t \le T$, the conditional probability $\mathbb{P}_{\rho 0}^T\left(G|\bar{\mathcal{G}}_s^0\right)$ of an event $G \in \bar{\mathcal{G}}_t^s$ can be computed using the POM E_t^s defined by (2.14) and just $\rho(s)$ as the conditional state of the system at time s. Indeed, taken $G \in \bar{\mathcal{G}}_t^s$, for all $\bar{\mathcal{G}}_s^0$-measurable random variables Y we have

$$\begin{aligned}\mathbb{E}_{\rho_O}^T\left[1_G Y\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\{\sigma(t)\}1_G Y\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathbb{E}_{\mathbb{Q}}\left[1_G\mathcal{A}(t,s)|\,\bar{\mathcal{G}}_s^0\right]\sigma(s)\right\}Y\right] \\ &= \mathbb{E}_Q\left[\mathrm{Tr}\left\{\mathbb{E}_Q\left[1_G\mathcal{A}(t,s)\right]\sigma(s)\right\}Y\right] = \mathbb{E}_Q\left[\mathrm{Tr}\left\{\mathbb{E}_Q\left[1_G\mathcal{A}(t,s)^*[\mathbb{I}]\right]\sigma(s)\right\}Y\right] \\ &= \mathbb{E}_Q\left[\mathrm{Tr}\left\{E_t^s(G)\sigma(s)\right\}Y\right] = \mathbb{E}_{\rho_0}^T\left[\mathrm{Tr}\left\{\rho(s)E_t^s(G)\right\}Y\right]\end{aligned}$$

This proves that $\mathrm{Tr}\left\{\rho(s)E_t^s(G)\right\}$ is the conditional expectation of 1_G given $\bar{\mathcal{G}}_s^0$. So, we have: $\forall G \in \bar{\mathcal{G}}_t^s, 0 \le s \le t \le T$,

$$\mathbb{P}_{\rho_0}^T\left(G|\bar{\mathcal{G}}_s^0\right) = \mathrm{Tr}\left\{\rho(s)E_t^s(G)\right\}$$

The random state $\rho(t)$ is called *a posteriori* state.

The mean of the *a posteriori* state

$$\eta(t) := \mathbb{E}_{\rho_0}^T[\rho(t)] = \mathbb{E}_{\mathbb{Q}}[\sigma(t)]$$

is the state to be assigned to the system when the result of the observation is not known or not taken into account; it is known as *a priori* state. We have

$$\eta(0) = \rho_0, \quad \eta(t) = \mathcal{T}(t,0)\,[\rho_0]$$

By using the composition property $\mathcal{A}(t,0) = \mathcal{A}(t,s) \circ \mathcal{A}(s,0)$ and the fact that $\mathcal{A}(t,s)$ and $\mathcal{A}(s,0)$ are $\mathbb{Q}$-independent, we obtain for $0 \le s < t \le T$

$$\mathbb{E}_{\mathbb{Q}}\,[\sigma(t)|\mathcal{F}_s] = \mathcal{T}(t,s)[\sigma(s)], \quad \mathbb{E}^T_{\rho_0}\,[\rho(t)|\mathcal{F}_s] = \mathcal{T}(t,s)[\rho(s)].$$

By differentiating (2.7) we get a stochastic evolution equation for the *a posteriori* states, known in the physical literature as stochastic master equation.

Proposition 2.7. *Under the physical probability $\mathbb{P}^T_{\rho_0}$, the a posteriori states satisfy the nonlinear SDE*

$$\mathrm{d}\rho(t) = \mathcal{L}(t)[\rho(t)]\,\mathrm{d}t + \sum \left[R_j(t)\rho(t) + \rho(t)R_j(t)^* - v_j(t)\rho(t)\right]\,\mathrm{d}\widehat{W}_j(t) \tag{2.17}$$

with initial condition $\rho(0) = \rho_0 \in \mathcal{S}(\mathcal{H})$. The quantities $v_j(t)$ are real random variables which depend on $\rho(t)$ and are given by (2.6).

Proof. By using (2.3) and (2.16) we can express the stochastic differential of $\sigma(t)$ in terms of the new noise $\widehat{W}$:

$$\mathrm{d}\sigma(t) = \mathcal{L}(t)[\sigma(t)]\,\mathrm{d}t + \sum_{j=1}^m \left(R_j(t)\sigma(t) + \sigma(t)R_j(t)^*\right)\,\mathrm{d}\widehat{W}_j(t)$$
$$+ \sum_{j=1}^m \left(R_j(t)\sigma(t) + \sigma(t)R_j(t)^*\right) v_j(t)\,\mathrm{d}t$$

From formula (2.5) we have immediately

$$(\mathrm{Tr}\{\sigma(t)\})^{-1} = \exp\left\{-\sum_{j=1}^m \left[\int_0^t v_j(s)\,\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t v_j(s)^2\,\mathrm{d}s\right]\right\}$$
$$= \exp\left\{-\sum_{j=1}^m \left[\int_0^t v_j(s)\,\mathrm{d}\widehat{W}_j(s) + \frac{1}{2}\int_0^t v_j(s)^2\,\mathrm{d}s\right]\right\}$$

by Itô formula we get

$$\mathrm{d}(\mathrm{Tr}\{\sigma(t)\})^{-1} = -(\mathrm{Tr}\{\sigma(t)\})^{-1}\sum_{j=1}^m v_j(t)\,\mathrm{d}\widehat{W}_j(t)$$

Finally, by Itô formula for products we get (2.17). □

Let us stress that our starting point was the linear SDE (2.3) for $\sigma(t)$ in the stochastic basis $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t\right), \mathbb{Q}\right)$. Then, we constructed the a posteriori states $\rho(t)$ by (2.7) and the stochastic basis $\left(\Omega, \mathcal{G}, \left(\bar{\mathcal{G}}_t^0\right), \mathbb{P}_{\rho_0}^T\right)$. Finally, we showed that, in this new stochastic basis, $\rho(t)$ satisfies (2.17). So, we have by construction that the nonlinear SDE (2.17) has a solution in a particular stochastic basis, the by Definition 1.35 we have shown that (2.17) has a weak solution. If the SDE (2.17) is extended from $\mathcal{S}(\mathcal{H})$ to the whole M_n then it has strong solutions, see Chapter 5 in [1] for more details.

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