Jianyu MA's DM Topology

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Soit $P(z,w)$ un polynôme en deux variables complexe, qu'on pensera comme une fonction $\mathbb{C}^2 \to \mathbb{C}$. Soit $\frac{\partial}{\partial z}, \frac{\partial}{\partial w}$ les dérivées standard (i.e. $\frac{\partial}{\partial z}\left(z^a w^b\right) = az^{a-1}w^b, \frac{\partial}{\partial w}\left(z^a w^b\right) = bz^a w^{b-1}$). Posons $z = x+iy$ et $w = u+iv$ avec $x,y,u,v \in \mathbb{R}$; on peut alors voir P aussi comme une fonction $(\Re(P), \Im(P)) : \mathbb{R}^4 \to \mathbb{R}^2$ ou $\Re(P)$ et $\Im(P)$ sont respectivement la partie réelle et la partie imaginaire de P. Pour toute fonctions $f,g : \mathbb{R}^4 \to \mathbb{R}$ soit alors $\frac{\partial}{\partial x}(f(x,y,u,v) + ig(x,y,u,v)) = \frac{\partial f(x,y,u,v)}{\partial x} + i\frac{\partial g(x,y,u,v)}{\partial x}$ et de facon similaire pour $\frac{\partial}{\partial y}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}$.

Question 1. *Prouver que* $\frac{\partial}{\partial z}\left(z^a w^b\right) = \frac{1}{2}\left(\frac{\partial}{\partial x}\left(z^a w^b\right) - i\frac{\partial}{\partial y}\left(z^a w^b\right)\right), \forall a,b \in \mathbb{N}$. *En conclure que l'on a* $\frac{\partial}{\partial z}(P) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(P)$. *Prouver aussi que* $\frac{\partial}{\partial x}\left(z^a w^b\right) + i\frac{\partial}{\partial y}\left(z^a w^b\right) = 0, \forall a,b \in \mathbb{N}$ *et en déduire que* $\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(P) = 0$

My Solution. $\forall a,b \in \mathbb{N}$,

$$\begin{aligned}
\frac{1}{2}\left(\frac{\partial}{\partial x}\left(z^a w^b\right) - i\frac{\partial}{\partial y}\left(z^a w^b\right)\right) &= \frac{1}{2}\left(\frac{\partial}{\partial x}\left((x+iy)^a w^b\right) - i\frac{\partial}{\partial y}\left((x+iy)^a w^b\right)\right) \\
&= \frac{1}{2}\left(a(x+iy)^{a-1}w^b - i * ia(x+iy)^{a-1}w^b\right) \\
&= a(x+iy)^{a-1}w^b = az^{a-1}w^b,
\end{aligned}$$

hence $\frac{\partial}{\partial z}(P) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(P)$.

By the same calculation,

$$\begin{aligned}
\frac{1}{2}\left(\frac{\partial}{\partial x}\left(z^a w^b\right) + i\frac{\partial}{\partial y}\left(z^a w^b\right)\right) &= \frac{1}{2}\left(\frac{\partial}{\partial x}\left((x+iy)^a w^b\right) + i\frac{\partial}{\partial y}\left((x+iy)^a w^b\right)\right) \\
&= \frac{1}{2}\left(a(x+iy)^{a-1}w^b + i * ia(x+iy)^{a-1}w^b\right) = 0,
\end{aligned}$$

we get $\frac{1}{2}\left(\frac{\partial}{\partial x}\left(z^a w^b\right) + i\frac{\partial}{\partial y}\left(z^a w^b\right)\right) = 0$, which means $\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(P) = 0$. From now on, we define $\frac{\partial}{\partial \bar{z}}(P) := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(P)$.

Question 2. *Prouver que si* $\frac{\partial}{\partial z}P(z_0,w_0) \neq 0$ *ou* $\frac{\partial}{\partial w}P(z_0,w_0) \neq 0$ *alors l'application* $(\Re(P), \Im(P)) : \mathbb{R}^4 \to \mathbb{R}^2$ *a jacobienne de rang 2 en* (z_0,w_0).

My Solution. By symmetry, we can assume W.L.O.G. that $\frac{\partial}{\partial z}P\left(z_0, w_0\right) \neq 0$, at this point the Jacobian of $(\Re(P), \Im(P))$ is

$$\mathrm{Jb}\left(\Re(P), \Im(P)\right)\big|_{(z_0,w_0)} = \begin{bmatrix} \frac{\partial}{\partial x}\Re(P) & \frac{\partial}{\partial y}\Re(P) & \dots \\ \frac{\partial}{\partial x}\Im(P) & \frac{\partial}{\partial y}\Im(P) & \dots \end{bmatrix}\Bigg|_{(z_0,w_0)},$$

and the determinant of the first 2×2 minor is

$$\begin{aligned}
\begin{vmatrix} \frac{\partial}{\partial x}\Re(P) & \frac{\partial}{\partial y}\Re(P) \\ \frac{\partial}{\partial x}\Im(P) & \frac{\partial}{\partial y}\Im(P) \end{vmatrix}_{(z_0,w_0)} &= \begin{vmatrix} (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})\Re(P) & \frac{\partial}{\partial y}\Re(P) \\ (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})\Im(P) & \frac{\partial}{\partial y}\Im(P) \end{vmatrix}_{(z_0,w_0)} \\
&= -i\begin{vmatrix} (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(P) & i\frac{\partial}{\partial y}(P) \\ (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})\Im(P) & i\frac{\partial}{\partial y}\Im(P) \end{vmatrix}_{(z_0,w_0)} \\
&= -i\begin{vmatrix} (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(P) & \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})(P) \\ (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})\Im(P) & \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})\Im(P) \end{vmatrix}_{(z_0,w_0)} \\
&= -i\begin{vmatrix} 2\frac{\partial}{\partial z}(P) & \frac{\partial}{\partial \bar{z}}(P) \\ 2\frac{\partial}{\partial z}\Im(P) & \frac{\partial}{\partial \bar{z}}\Im(P) \end{vmatrix}_{(z_0,w_0)} \\
&= \begin{vmatrix} \frac{\partial}{\partial z}(P) & \frac{\partial}{\partial \bar{z}}(P) \\ -2i\frac{\partial}{\partial z}\Im(P) & -2i\frac{\partial}{\partial \bar{z}}\Im(P) \end{vmatrix}_{(z_0,w_0)} \\
&= \begin{vmatrix} \frac{\partial}{\partial z}(P) & \frac{\partial}{\partial \bar{z}}(P) \\ \frac{\partial}{\partial z}(\bar{P}) & \frac{\partial}{\partial \bar{z}}(\bar{P}) \end{vmatrix}_{(z_0,w_0)} \\
&= \left|\frac{\partial}{\partial z}(P)\right|^2_{(z_0,w_0)} - \left|\frac{\partial}{\partial \bar{z}}(P)\right|^2_{(z_0,w_0)} = \left|\frac{\partial}{\partial z}P(z_0,w_0)\right|^2.
\end{aligned}$$

So this Jacobian is of rank 2.

Question 3. *Soit* $S = \{(z,w)|P(z,w) = 0\}$.

a) *Prouver que* S *est aussi Haussdorf et à base dénombrable.*

b) *Prouver que si* S *ne contient aucun point tel que* $\frac{\partial}{\partial z}P(z,w) = \frac{\partial}{\partial w}P(z,w) = 0$ *alors c'est un espace localement homeomorphe à* $\mathbb{R}^2$.

c) *De plus montrer que si* $\frac{\partial}{\partial w}P\left(z_0, w_0\right) \neq 0$ *alors la restriction de la projection* $\pi_1(z,w) = z$ *à* S *est un homéomorphisme dans un voisinage de* (z_0, w_0).

My Solution. a) $\mathbb{C}^2$ is a separable metric space, hence is Hausdorff and has a countable topology basis. S is a subspace of $\mathbb{C}^2$ so it is Hausdorff and has a countable topology basis.

b) If the set $\{(z,w) \in \mathbb{C}^2 | \frac{\partial}{\partial z}P(z,w) = \frac{\partial}{\partial w}P(z,w) = 0\} = \emptyset$, then from the last question the Jacobian of $(\Re(P), \Im(P))$ is always of constant rank 2. So S is a smooth manifold of dimension $\dim(\mathbb{C}^2) - \mathrm{rank}(\mathrm{Jb}(\Re(P), \Im(P))) = 2$ and is locally homeomorphic to $\mathbb{R}^2$.

c) If $\frac{\partial}{\partial w} P(z_0, w_0) \neq 0$, since the last 2×2 minor of $\mathrm{Jb}\left(\Re(P), \Im(P)\right)|_{(z_0,w_0)}$ is $\left|\frac{\partial}{\partial w} P(z_0, w_0)\right|^2$ then by inverse function theorem there is an open neighborhood $U(x_0, y_0)$ of z_0 and $\exists f \in C^\infty : U(x_0, y_0) \to \mathbb{R}^2$ such that ($\mathbb{C}$ is identified with $\mathbb{R}^2$)

$$\begin{aligned} S \cap (U(x_0, y_0) \times \mathbb{C}) &= \{(x, y, u, v) \in U(x_0, y_0) \times \mathbb{R}^2 | P(z, w) = 0\} \\ &= \{(x, y, f(x, y)) | (x, y) \in U(x_0, y_0)\} \\ &= \{(z, f(z) | z \in U(x_0, y_0)\} \end{aligned}$$

So $\pi_1|_{S \cap (U(x_0,y_0) \times \mathbb{C})} : (z, f(z) \mapsto z$ is a homeomorphism.

Question 4. *Soit maintenant $P(z, w) = w^2 - z(z-1)(z-2)$. Montrer que dans ce cas S satisfait l'hypothèse du point 3b).*

My Solution. The set of critical points of P is

$$\begin{aligned} \mathrm{Crit}(P) &= \{(z, w) \in \mathbb{C}^2 | \frac{\partial}{\partial z} P(z, w) = \frac{\partial}{\partial w} P(z, w) = 0\} \\ &= \{(z, w) \in \mathbb{C}^2 | w = 3z^2 - 6z - 2 = 0\} = \{(1 \pm \sqrt{\frac{5}{3}}, 0)\} \end{aligned}$$

then easily we can check that $\mathrm{Crit} \cap S = \emptyset$. So condition b) in the last question is satisfied.

Question 5. *For $\varepsilon \in]0, 1/10[$ let $C \subset \mathbb{C}$ be defined by $C = B(0, 100) \setminus$ int $(B(0, \epsilon) \cup B(1, \epsilon) \cup B(2, \epsilon))$ where $B(x, r)$ is the ball centered in x and of radius r and int is the interior. Let $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ be the projection on the first coordinate coordonnée : $\pi_1(z, w) = z$ and let $S_C = S \cap \pi_1^{-1} C$. Show that $\pi_1 : S_C \to C$ is a covering of C. What is its degree?*

My Solution. Fix a $(z_0, w_0) \in \pi_1^{-1}(C)$, consider two maps

$$\begin{cases} u : & z \in \pi_1(C) & \mapsto z(z-1)(z-2) \\ v : & w \in \mathrm{img}(u) & \mapsto w^2 \end{cases}$$

and we want to apply the inverse function theorem to u and v. The critical points set of u is $\mathrm{Crit}(u) = \pi_1(\mathrm{Crit}(P)) \cap \pi_1(C) = \emptyset$ and for v we have $\mathrm{Crit}(v) = \pi_2(\mathrm{Crit}(P)) \cap \mathrm{img}(u) = \{0\} \cap \mathrm{img}(u) = \emptyset$. Apply inverse function theorem firstly to v then to u we can get four smooth homemorphisms

$$\begin{aligned} v : V_{10} \to V_0 \quad & v : V_{11} \to V_0 \\ u : U_{10} \to V_{10} \quad & u : U_{11} \to V_{11} \end{aligned}$$

where V_0 is an open neighborhood of w_0, and $V_{10} = -V_{11}, V_{10} \cap V_{11} = \emptyset, z_0 \in U_{10} \cap U_{11}$. We define $U := U_{10} \cap U_{11}$, since $P(z, w) = u(z) - v(w)$ we know that there are two disjoint components in $\pi_1^{-1}(U) \cap S_C$, separately contained in $U \times V_{10}$ and $U \times V_{11}$. Moreover, $\pi_1 : (U \times V_{10}) \cap S_C \to U$ and $\pi_1 : (U \times V_{11}) \cap S_C \to U$ are homeomorphisms. Hence we prove that $\pi_1 : S_C \to C$ is a 2 degree covering of C.

Question 6. *Let $S^i := S \cap \pi_1^{-1}(B(i,\epsilon)), i \in \{0,1,2\}$. Show that if ϵ is sufficiently small the projection $\pi_2 : S^i \to \mathbb{C}$ defined by $\pi_2(z,w) = w$ is a homeomorphism. Deduce that S^i is diffeomorphic to a disc.*

My Solution. Let's fix an $i \in \{0,1,2\}$ then we have $(i,0) \in S$. If ϵ is small enough then $\frac{\partial}{\partial z}P \neq 0$ in S^i, so as the proof in 3*c*) we can find an open neighborhood X_i of $(i,0)$ in which π_2 is a homeomorphism. Since $\pi_2(X_i)$ is a neighborhood of $(i,0)$ we can set a ϵ such that $(B(i,\varepsilon)) \in \pi_2(X_i)$. And in this case $\pi_2 : S^i \to \mathbb{C}$ is a homeomorphism onto its image.

S^i is diffeomorphic to the disk $B(i,\epsilon)$ because $\pi_2(S^i) = B(i,\epsilon)$ and both π_2 and its inverse are sooth functions.

Question 7. *Montrer que S_C est connexe.*

My Solution. S_C is locally connected so its connected components are closed and open. We claim that the image under π_1 of each component in S_C is C. Otherwise if a component $A \subset S_C$ satisfies $\pi_1(A) \neq C$, let O_a be the fundamental neighborhood of $a \in \pi_1(\partial A)$. Then the part of $\pi_1^{-1}(O_a)$ that intersects A should be contained in A, thus $a \in \pi_1(\mathring{A})$ since π_1 is a local homeomorphism and hence we get a contradiction as $\mathring{A} \cap \partial A = \emptyset$. In addition S_C is a degree 2 covering space, it has at most two components.

Now we assume that S_C has exact two components A_1, A_2. Then each fiber $\{a_1, a_2\}$ of a singleton $\pi_1(a_1) = \pi_1(a_2)$ in C lies in two components separately and $\pi_2(a_1) = -\pi_2(a_2)$. Let's recall the definition of $u : z \in C \mapsto z(z-1)(z-2)$, plotted as Figure 1 and meshed with contour line. We shouldn't have closed contour circle in the plot of $|u(z)|$ otherwise if $\exists \rho \in \mathbb{R}^*$, s.t. $\{\rho^2 e^{i\theta}, \theta \in \mathbb{R}\} \subset u(C)$ then (ρ^2, ρ) and $(\rho^2, -\rho)$, which should be in two different components, are connected by path γ

$$\begin{aligned} \gamma : [0,1] &\to S_C \\ t &\mapsto (\rho^2 e^{i2t\pi},\ \rho e^{it\pi}). \end{aligned}$$

Let $m := \sup\{|u(z)|, z \in B(0,2\epsilon) \cup B(1,2\epsilon) \cup B(2,2\epsilon)\}, M := \inf\{|u(z)|, |z| \geq 99\}$ then we have $m < M$ and any contour line with value between m and M is closed in the plot of $u(z)$, so we finally get a contradiction.

Question 8. *Montrer que si un espace $p : Y \to X$ est un revêtement et X est un CW-complexe, alors Y peut être muni d'une structure de CW -complexe telle que chaque cellule de X est l'image par p d'au moins une cellule de Y. Combien de cellules de chaque dimension a Y?*

My Solution. For clarity, let's recall definitions of some terms. Let $K^{(0)}$ be a discrete set of points. These points are the 0-cells. If $K^{(n-1)}$ has been defined, let $\{f_{\partial\sigma}\}$ be a collection of maps $f_{\partial\sigma} : \mathbf{S}^{n-1} \to K^{(n-1)}$ where σ ranges over some indexing set. Let W be the disjoint union of copies $\mathbf{D}^n_\sigma$ of $\mathbf{D}^n$, one for each σ, let B be the corresponding union of the boundaries $\mathbf{S}^{n-1}_\sigma$ of these disks, and put together the maps $f_{\partial\sigma}$ to produce a map $f : B \to K^{(n-1)}$. Then define

$$K^{(n)} = K^{(n-1)} \cup_f W.$$

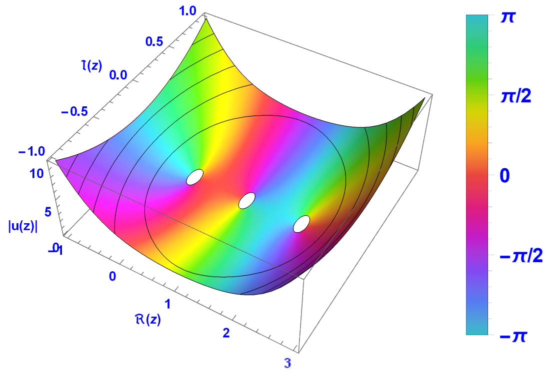

Figure 1: Complex plot of $u : z \in C \mapsto z(z-1)(z-2)$ with $\epsilon = 0.1$

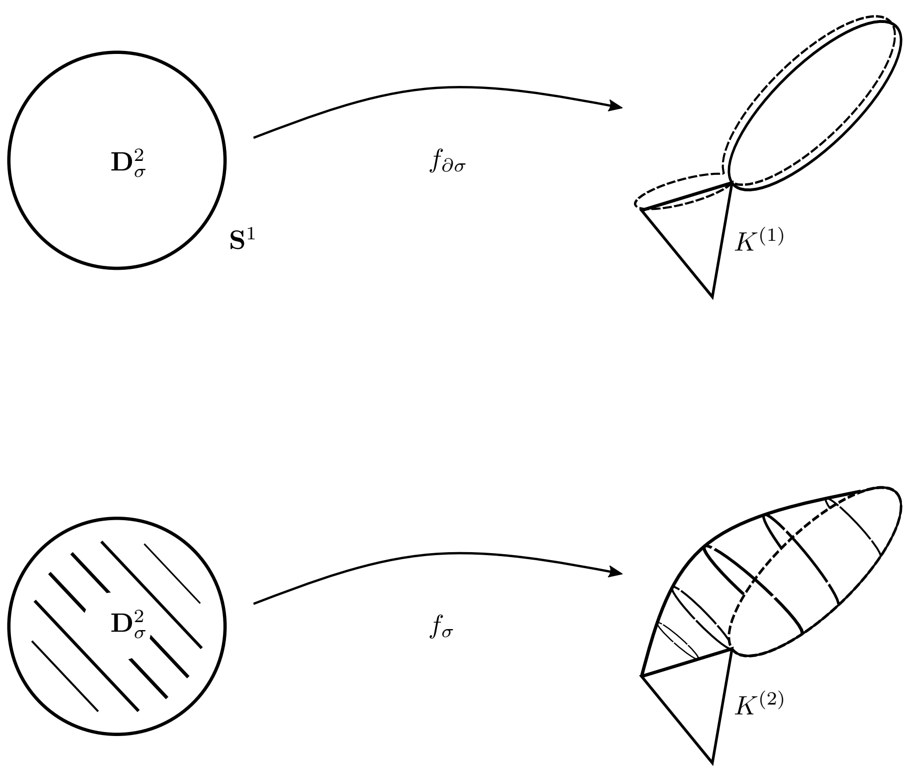

Figure 2: Attaching map and characteristic map

The map $f_{\partial\sigma}$ is called the "attaching map" for the cell σ.

If $K^{(n)}$ has been defined for all integers $n \geq 0$, let $K = \bigcup K^{(n)}$ with the "weak" topology that specifies that a set is open $\Leftrightarrow$ its intersection with each $K^{(n)}$ is open in $K^{(n)}$. (It follows that a set is closed $\Leftrightarrow$ its intersection with each $K^{(n)}$ is closed.) For each σ let $f_\sigma : \mathbf{D}^n_\sigma \to K$ be the canonical map given by the attaching of the cell σ. This map is called the "characteristic map" of the cell σ. Let K_σ be the image of f_σ. See Figure 2.

It is clear that the topology of each $K^{(n)}$, and hence of K itself, is characterized by the statement that a subset is open (closed) $\Leftrightarrow$ its inverse image under each f_σ is open (closed) $\Leftrightarrow$ its intersection with each K_σ is open (closed) in K_σ where the topology of the latter is the topology of the quotient of $\mathbf{D}^n$ by the identifications made by the attaching map $f_{\partial\sigma}$.

For our proposition, let $p : Y \to X$ be a covering map and assume that X is a CW-complex with characteristic maps $f_\alpha : \mathbf{D}^n \to X$. Since $\mathbf{D}^n$ is simply connected, each f_α lifts to maps $f_{\tilde{\alpha}} : \mathbf{D}^n \to Y$ which are unique upon specification of the image of any point. Take the collection of all such liftings of all f_α to define a cell structure on Y. That is to say, in each dimension of skeleton, there are as n times cells in Y as in X, where n is the degree of this covering space.

Then the only thing that really needs proving is that Y has the weak topology. That is, we must show that a set $A \subset Y$ is open $\Leftrightarrow$ each $f_{\tilde{\alpha}}^{-1}(A)$ is open in the disk which is the domain of $f_{\tilde{\alpha}}$. The implication $\Rightarrow$ is trivial since $f_{\tilde{\alpha}}$ is continuous. Thus we must show that if $A \subset Y$ has each $f_{\tilde{\alpha}}^{-1}(A)$ open, then A is open. If U ranges over all components of $p^{-1}(V)$ where V ranges over all connected evenly covered open sets in X, then $A = \bigcup(A \cap U)$ and $f_{\tilde{\alpha}}^{-1}(A \cap U) = f_{\tilde{\alpha}}^{-1}(A) \cap f_{\tilde{\alpha}}^{-1}(U)$. This shows that it suffices to consider the case in which $A \subset U$ for some such U. We claim that

$$f_\alpha^{-1}(p(A)) = \bigcup \left\{ f_{\tilde{\alpha}}^{-1}(A) | f_{\tilde{\alpha}} \text{ a lift of } f_\alpha \right\}$$

Indeed, if $x \in f_\alpha^{-1}(p(A))$ then $f_\alpha(x) = p(a)$ for some $a \in A$ and there exists a lifting $f_{\tilde{\alpha}}$ of f_α such that $f_{\tilde{\alpha}}(x) = a$. Thus $x \in f_{\tilde{\alpha}}^{-1}(a) \subset f_{\tilde{\alpha}}^{-1}(A)$. Conversely, if $x \in f_a^{-1}(A)$ then $f_a(x) = a \in A$ and so $f_a(x) = (p \circ f_{\tilde{a}})(x) = p(a) \in p(A)$, giving that $x \in f_a^{-1}(p(A))$, as claimed. Therefore, if $f_{\tilde{a}}^{-1}(A)$ is open for all $\tilde{\alpha}$, then the union above is open and so $f_a^{-1}(p(A))$ is open for all α. since X has the weak topology by definition, $p(A)$ is open. But $A \subset U$ and $p : U \to p(U) = V$ is a homeomorphism by the assumption that U is a component of $p^{-1}(V)$ for the evenly covered open set V. Therefore, A is open in U and hence in Y.

Question 9. *Considérons la structure de CW-complexe de C ayant 8 0-cellules, 12 1-cellules et 2 2-cellules, comme dans la Figure 3 . En appliquant la construction du point précédent, constuire une structure de CW complexe sur S_C. Combien de 0,1 et 2 -cellules a cette cellularization de S ? (On pourra utiliser le théorème de relèvement des applications.)*

My Solution. A bit hard to image S_C as CW-Complex, maybe it is not possible to be embedded into $\mathbb{R}^3$; a plot of $\Im(w)$ is shown in Figure 4, just as the Klein

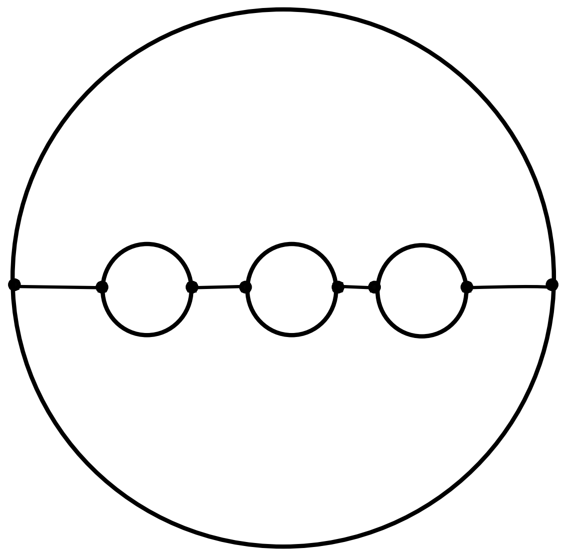

Figure 3: Space C

this projection to $\Im(w)$ intersects itself. To construct a CW-complex, we use the lifting method described in previous solution. This lifting is similar to lifting $\sqrt{z}$ in complex plane, but I cannot draw it out explicitly. Since each cell of C is the homeomorph under a 2 degree covering map π_1 of a cell in S_C, for each dimension in S_C there should be as twice cells as in C. Thus this cellularization of S_C contains 16 0-cells, 24 1-cells and 4 2-cells.

Question 10. *Etant donnée une structure de CW -complexe sur un espace X, ayant un nombre fini de cellules, sa caractéristique d'Euler est $\chi(X) := \sum_i (-1)^i c_i$ où c_i est le nombre de cellules de dimension i. Calculer $\chi(C)$ et $\chi(S_C)$.*

My Solution. $\chi(C) = 8 - 12 + 2 = -2$ and $\chi(S_C) = 2\chi(C) = -4$

Question 11. *Soit X un CW-complexe fini dont la dimension maximale des cellules est 2. Une subdivision de la structure de CW-complexe est une structure de CW-complexe obtenue de la première en appliquant un nombre fini des modifications suivantes (cf Figure 5):*

- *Subdiviser une 1-cellule : Ajouter une 0-cellule au milieu d'une 1-cellule et remplaçer la 1-cellule par deux 1 -cellules.*
- *Subdiviser une 2-cellule : Ajouter une 0-cellule au milieu d'une 2-cellule, des 1-cellules reliant cette 0-cellules à toutes les 0-cellules dans son bord et remplacer la 2-cellule par des 2-cellules (une par 1-cellule ajoutée).*

Montrer que la caractéristique d'Euler d'une subdivision coincide avec la caractéristique d'Euler de la structure initiale.

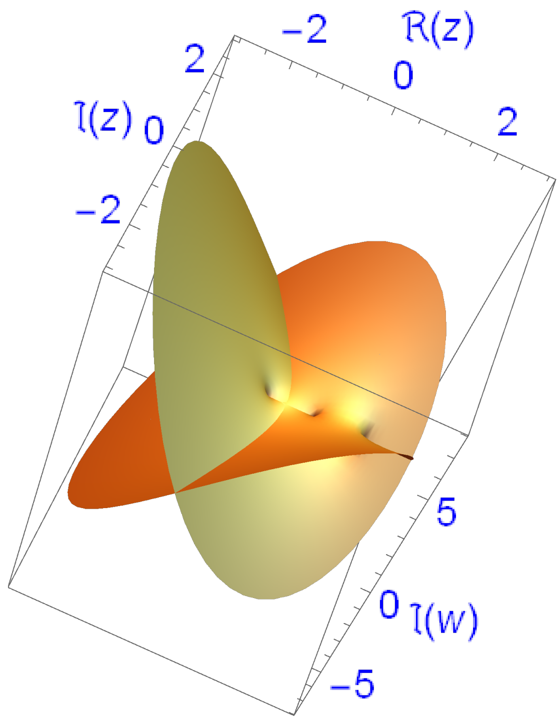

Figure 4: Project Riemann surface S_C to $\Im(w)$

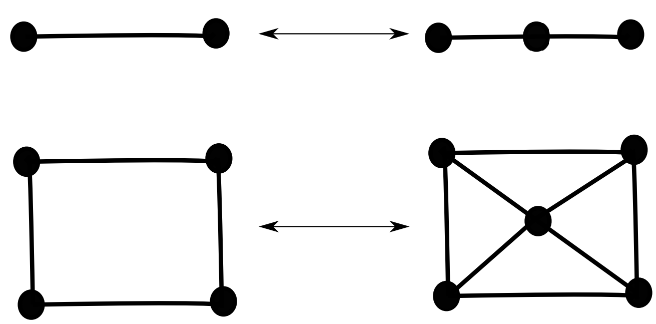

Figure 5: On the top part a subdivision of a 1-cell. On the bottom a subdivision of a 2-cell whose boundary contains 4 0-cells.

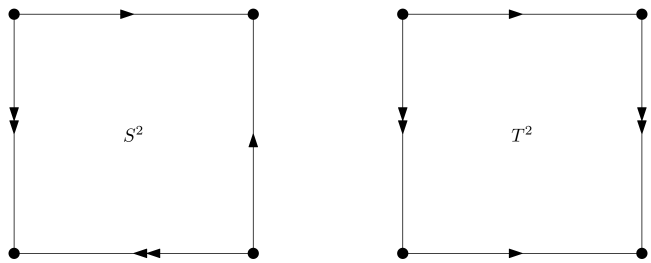

Figure 6: Represent S^2 and T^2 as quotient spaces of unit square

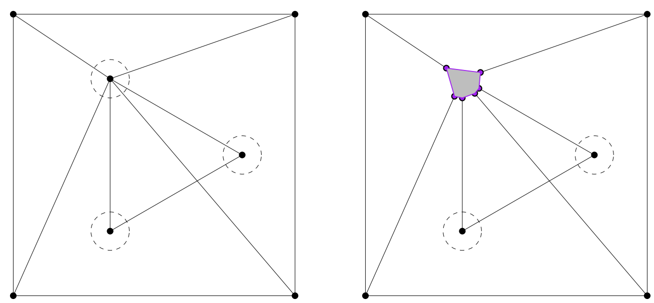

Figure 7: Puncture a hole in the square

My Solution.

- After the first operation, we get 1 more 0-cell, 1 more 1-cell, and others remain unchanged. These changes cancel each other in Euler characteristic as $(c_0 + 1) - (c_1 + 1) = c_0 - c_1$.
- If we divide a 2-cell with n 0-cells in boundary using the second operation, we get 1 more 0-cell, n more 1-cells, $n - 1$ more 2-cells, and others remain unchanged. These changes cancel each other in Euler characteristic as $(c_0 + 1) - (c_1 + n - 1) + (c_2 + n) = c_0 - c_1 + c_2$.

Question 12. *Pour tout $n \geqslant 0$ calculer la caractéristique d'Euler de $S^2 \setminus \left(\mathring{D}_1^2 \sqcup \mathring{D}_2^2 \sqcup \cdots \sqcup \mathring{D}_n^2\right)$ et de $T^2 \setminus \left(\mathring{D}_1^2 \sqcup \mathring{D}_2^2 \sqcup \cdots \sqcup \mathring{D}_n^2\right)$ où $\left(\mathring{D}_1^2 \sqcup \mathring{D}_2^2 \sqcup \cdots \sqcup \mathring{D}_n^2\right)$ discs sont $n \geqslant 0$ discs ouverts et disjoints contenus dans S ou T.*

My Solution. As Figure 6, we represent S^2 and T^2 as quotient spaces of unit square. Then by counting different vertices and edges after gluing, we have

$$\chi(S^2) = c_0 - c_1 + c_2 = 3 - 2 + 1 = 2$$
$$\chi(T^2) = c_0 - c_1 + c_2 = 1 - 2 + 1 = 0.$$

To calculate Euler characteristic of n disks punctured S^2 or T^2, we turn to consider punctured square, as Figure 7. Add a regular n-polygon in the center of square, connect each vertex of n-polygon with all vertices of that square. Then replace vertex in the regular n-polygon with an irregular 6-polygon for which we delete the interior part. And we color new generated vertices and edges in purples.

Denote S^2_n and T^2_n corresponding n-disks punctured S^2_n and T^2_n. After identifying some vertices in square, we calculate

$$\begin{aligned}\chi(S^2_n) &= c_0' - c_1' + c_2' = (c_0 + n(-1 + c_0 + 2)) - (c_1 + n(c_0 + 2)) + c_2 \\ &= (3 + 4n) - (2 + 5n) + 1 = 2 - n \\ \chi(T^2_n) &= c_0' - c_1' + c_2' = (c_0 + n(-1 + c_0 + 2)) - (c_1 + n(c_0 + 2)) + c_2 \\ &= (1 + 4n) - (2 + 5n) + 1 = -n.\end{aligned}$$

Question 13. *La surface à bord S_C est homéomorphe à $X \backslash \left(\mathring{D}^2_1 \sqcup \mathring{D}^2_2 \sqcup \cdots \sqcup \mathring{D}^2_n\right)$ où X est l'une des surfaces S^2 ou T^2. Peut-on déterminer laquelle en utilisant uniquement la caractéristique d'Euler?*

My Solution. No, we cannot because we have $\chi(S_C) = -4 = \chi(S^2_6) = \chi(T^2_4)$.

Question 14. *Révisez la notion de variété à bord. On remarque que S_C et C sont des variétés de dimension 2 à bord et que $\pi_1 : S_C \to C$ est une application différentiable. Compter le nombre de composantes de bord de S_C (i.e. le nombre de composantes connexes de ∂S_C). Répondre alors au point précédent.*

My Solution. To see that S_C has smooth boundary, we can enlarge the outer radius and reduce the value of ϵ in C then the boundary of S_C can be seen as the locally homeomorphism preimage of smooth circles hence smooth as well. $\pi_1 : S_C \to C$ is the restriction to a regular submanifold of a smooth map between two manifolds $\mathbb{C}^2$ and $\mathbb{C}$ and hence smooth again.

The preimage of a component in ∂C is connected in S_C since here we can topologically view this covering map as $p : z \in S^1 \mapsto z^2 \in S^1$ in complex plane. Therefore, there are four components in ∂S_C and for previous question we then know S_C is homeomorphic to $T^2_4 := T^2 \backslash \left(\mathring{D}^2_1 \sqcup \mathring{D}^2_2 \sqcup \mathring{D}^2_3 \sqcup \mathring{D}^2_4\right)$.

Question 15. *Si on considère $S_C \cup S^0 \cup S^1 \cup S^2$ alors il s'agit d'une variété de dimension 2 homéomorphe à une de la liste ci-dessus. Laquelle?*

My Solution. We have four homeomorphisms

$$\begin{aligned}\pi_1 :& S_C \cong T^2 \backslash \left(\mathring{D}^2_1 \sqcup \mathring{D}^2_2 \sqcup \mathring{D}^2_3 \sqcup \mathring{D}^2_4\right) \\ \pi_2 :& S^i \cong D(i, \epsilon), \text{ where } i \in \{1, 2, 3\},\end{aligned}$$

combine them we get $S_C \cup S^0 \cup S^1 \cup S^2 \cong T^2_1 := T^2 \backslash \mathring{D}^2_1$.